Differential Equations

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1 Daily Quiz

Find a factor μ that transforms the following to an exact differential equation:

$$\left(6ty^2 + 2y\right) + \left(12t^2y + 6t + 3\right) y' = 0.$$

2 Key Topics

Today we consider the existence and uniqueness of solutions to the first-order initial value problem:

$$y' = f(t, y),\ \ y(t_0) = y_0. \tag{1}$$

For further reading, see [2, Section 2.3] and [1, Section 1.6].

3 Existence and Uniqueness Theorem

The following theorem provides sufficient conditions for when a first-order initial value problem has a unique solution.

Theorem 3.1. *Let f and $\frac{\partial f}{\partial y}$ be continuous on the rectangle*

$$R = \{(t, y) \colon\ \ |t - t_0| \leq a,\ \ |y - y_0| \leq b\}$$

then the initial value problem (1) *has a unique solution $y(t)$ on the interval*

$$|t - t_0| \leq \min\left\{a, \frac{b}{M}\right\},$$

where $M = \max_R |f(t, y)|$.

While a proof of Theorem 3.1 is outside the scope of our course, we can demonstrate its sufficiency for guaranteeing the existence and uniqueness of a solution using the *method of successive approximations* or *Picard's method*.

Let $\phi_0(t)$ be any differentiable function of t that satisfies the initial condition in (1), i.e., $\phi_0(t_0) = y_0$. Then, for any integer $n \geq 1$, define $\phi_n(t)$ as the antiderivative of $f(t, \phi_{n-1}(t))$ that satisfies the initial condition in (1). This method of successive approximations constructs a sequence of functions

$$\phi_0(t), \phi_1(t), \phi_2(t), \ldots,$$

which can be used to prove Theorem 3.1 by showing the following

I. All members of the sequence $\{\phi_n(t)\}_{n=0}^{\infty}$ are well-defined

II. $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ exists.

III. $\phi(t)$ satisfies the initial value problem (1).

IV. $\phi(t)$ is the only solution to the initial value problem (1).

Example 3.2. Consider the initial value problem

$$y' = 2t(1+y), \; y(0) = 0.$$

Define $\phi_0(t) = 0$. Then, the method of successive approximations finds

$$\begin{aligned}
\phi_1(t) &= \int_0^t 2s(1+\phi_0(s))ds = \int_0^t 2sds = t^2 \\
\phi_2(t) &= \int_0^t 2s(1+s^2)ds = \int_0^t \left(2s + 2s^3\right) ds = t^2 + \frac{1}{2}t^4 \\
\phi_3(t) &= \int_0^t 2s\left(1 + s^2 + \frac{1}{2}s^4\right) ds = \int_0^t \left(2s + 2s^3 + s^5\right) ds = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 \\
&\vdots \\
\phi_n(t) &= t^2 + \frac{1}{2!}t^4 + \frac{1}{3!}t^6 + \cdots + \frac{1}{n!}t^{2n}.
\end{aligned}$$

Recall the Taylor series for e^t centered at 0:

$$e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots.$$

Hence,

$$\lim_{n\to\infty} \phi_n(t) = e^{t^2} - 1.$$

4 Exercises

Consider the initial value problem

$$y' = y^{2/3}, \; y(0) = 0.$$

Show that $y(t) = 0$ and $u(t) = \frac{1}{27}t^3$ are both solutions to the given IVP. Why does this example not violate Theorem 3.1.

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.