Differential Equations

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1 Daily Quiz

Does the following initial value problem have a unique solution?

$$y' = t + \arcsin(y),\ y(1) = 1.$$

Why or why not?

2 Key Topics

Today we solve first-order differential equations that arise from mathematical models of natural phenomena. In particular, we will study growth, decay, cooling, and mixing problems. For further reading, see [2, Chapter 4].

3 Growth and Decay

Let $Q := Q(t)$ denote the number of inhabitants in a population at time t. The Malthus model assumes that Q' is proportional to Q:

$$Q' = \lambda Q, \tag{1}$$

where λ is a constant of proportionality. The differential equation in (1) is a one-parameter family of first-order autonomous differential equations. Hence, we can analyze the solutions using a phase line. Also, we know that the general solution is given by $Q(t) = ce^{\lambda t}$.

The Verhulst model assumes that

$$Q' = k\left(1 - \frac{1}{N}Q\right)Q, \tag{2}$$

where k and N are constants. The differential equation in (2) is separable and therefore a general solution can be found using separation of variables.

4 Cooling and Mixing

Let $T := T(t)$ denote the temperature of an object at time t. If the object is in a medium of temperature $T_m(t)$, then Newton's law of cooling states that

$$T' = -k(T - T_m). \tag{3}$$

The differential equation in (3) is first-order linear. Hence, a general solution can be found using an integration factor.

Consider a 600 gallon tank that initially holds a saltwater solution with 40 pounds of salt. Suppose that water with a salt concentration 0.5 lb/gal enters the tank at a rate of 4 gal/min, and that the well-mixed solution leaves the tank at a rate of 4 gal/min. Let $Q := Q(t)$ denote the quantity of salt in the saltwater solution. Then, the rate of change of salt in the solution is given by

$$Q' \, \frac{lb}{min} = \left(0.5 \, \frac{lb}{gal}\right)\left(4 \, \frac{gal}{min}\right) - \left(\frac{Q}{600} \, \frac{lb}{gal}\right)\left(4 \, \frac{gal}{min}\right).$$

Therefore, $Q(t)$ is the unique solution of the initial value problem:

$$Q' = 2 - \frac{1}{150}Q, \; Q(0) = 40. \tag{4}$$

5 Exercises

I. Compare the general solution of the Malthus model to a phase line analysis.

II. Find the general solution of the Verhulst model in (2) using separation of variables.

III. Find a general solution of the Newton cooling model in (3) using an integration factor. You may assume that T_m is constant.

IV. Find the particular solution of the mixing initial value problem (4).

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.