Compactness in the Real Numbers

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1 From Finiteness to Compactness

Many arguments are easy for a finite set. If every point is assigned a useful neighborhood, only finitely many neighborhoods are needed because there are only finitely many points. An interval contains infinitely many points, so the same reasoning does not work directly. Compactness supplies a substitute: from an arbitrary collection of open sets covering the set, we may retain finitely many and still cover it.

This idea will become especially useful when we study sequences and continuous functions. Compactness will help us extract convergent subsequences, turn pointwise local information into uniform information, and show that continuous functions attain maximum and minimum values.

Definition 1.1. Let $S \subseteq \mathbb{R}$, and let $\mathcal{U}$ be a collection of sets. The collection $\mathcal{U}$ is a *cover* of S if

$$S \subseteq \bigcup_{U \in \mathcal{U}} U.$$

It is an *open cover* if every $U \in \mathcal{U}$ is open. If $\mathcal{V} \subseteq \mathcal{U}$ and $\mathcal{V}$ still covers S, then $\mathcal{V}$ is a *subcover of S selected from $\mathcal{U}$*. A subcover is *finite* if it contains only finitely many sets.

The sets in a cover need not be subsets of S, and their union need not equal S. They need only contain all points of S.

Example 1.2 (A first finite cover). The three open intervals

$$(-1, 2/3), \qquad (1/5, 4/5), \qquad (3/5, 2)$$

cover $[0, 1]$. The first and third intervals already form a finite subcover. Notice that the intervals extend outside $[0, 1]$; this is allowed. On the other hand, the first two intervals do not cover $[0, 1]$, because they miss every point from $[4/5, 1]$.

Example 1.3 (An infinite cover with a finite subcover). Let $S = [0, 1]$ and

$$\mathcal{U} = \{(-1, 3/4), (1/2, 2)\} \cup \{(1/(n+2), 1 - 1/(n+2)) : n \in \mathbb{N}\}.$$

This is an infinite open cover of S. The finite subcollection

$$\mathcal{V} = \{(-1, 3/4), (1/2, 2)\}$$

is still a cover of S, so $\mathcal{V}$ is a finite subcover of S selected from $\mathcal{U}$. Neither interval in $\mathcal{V}$ covers S alone: the first misses 1, while the second misses 0. Thus this example genuinely requires both selected intervals, even though the original open cover contains infinitely many additional sets.

Example 1.4 (A cover with no finite subcover). Let $S = (0, 1)$. For $n \geq 2$, define

$$U_n = (-1, 1 - 1/n), \qquad \mathcal{U} = \{U_n : n \geq 2\}.$$

The collection $\mathcal{U}$ is an open cover of S. Indeed, if $x \in (0, 1)$, then $1 - x > 0$. By the Archimedean property, we may choose $n \geq 2$ so large that $1/n < 1 - x$; hence $x < 1 - 1/n$, and therefore $x \in U_n$.

However, no finite subcollection covers S. Suppose $U_{n_1}, \ldots, U_{n_k}$ are selected, and let

$$N = \max\{n_1, \ldots, n_k\}.$$

Since the intervals are nested,

$$U_{n_1} \cup \cdots \cup U_{n_k} = U_N = (-1, 1 - 1/N).$$

The point $1 - 1/(2N)$ belongs to S but not to U_N. Thus this open cover of S has no finite subcover.

Definition 1.5. A set $S \subseteq \mathbb{R}$ is *compact* if, for every open cover $\mathcal{U}$ of S, there is a finite subcollection $\mathcal{V} \subseteq \mathcal{U}$ that still covers S.

Compactness does not say that every open cover is finite. It says that even when an open cover contains infinitely many sets, finitely many of the *original* sets are enough. We are not allowed to replace the sets by new ones while searching for the finite subcover.

The order of the quantifiers is important. To prove that S is compact, we must begin with an *arbitrary* open cover and find a finite subcover. To prove that S is not compact, it is enough to construct *one* open cover having no finite subcover.

Proposition 1.6. *Every finite subset of $\mathbb{R}$ is compact.*

Proof. Let $S = \{x_1, \ldots, x_n\}$, and let $\mathcal{U}$ be an open cover of S. For each i, choose $U_i \in \mathcal{U}$ with $x_i \in U_i$. Then $U_1, \ldots, U_n$ form a finite subcover. The empty set is also compact: the empty collection is a finite subcover of every cover of $\varnothing$. □

2 How Compactness Can Fail

Example 1.4 gives the first model for proving noncompactness: construct one open cover for which every finite subcollection leaves behind a point of the set. The next examples illustrate two other common obstructions.

Example 2.1 (An unbounded set). The collection $\mathcal{U} = \{(-n, n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$. Any finite subcollection has a largest index N, and its union is $(-N, N)$, which does not cover $\mathbb{R}$. Therefore $\mathbb{R}$ is not compact.

Example 2.2 (Bounded but not closed). The set $\{1/n : n \in \mathbb{N}\}$ is not compact. For $n \in \mathbb{N}$, define $U_n = (1/(n+1), 2)$. These open sets cover every $1/k$. A finite subcollection has a largest index N, and its union is U_N. But $1/(N+1) \notin U_N$, so no finite subcollection covers the set. The missing accumulation point 0 is the obstruction.

The examples suggest that compact subsets of $\mathbb{R}$ must be both bounded and closed. We now prove those two necessities directly from the definition.

Proposition 2.3. *Every compact subset of $\mathbb{R}$ is bounded.*

Proof. The empty set is bounded, so assume that $S \neq \varnothing$. The intervals $(-n, n)$, $n \in \mathbb{N}$, form an open cover of S. If S is compact, some finite subcollection covers S. Because S is nonempty, this subcollection contains at least one interval. Let N be its largest index. Because the intervals are nested, their union is $(-N, N)$. Thus $S \subseteq (-N, N)$, so S is bounded. □

Proposition 2.4. *Every compact subset of $\mathbb{R}$ is closed.*

Proof. The empty set is closed, so assume that $S \neq \varnothing$. Let S be compact and let $p \notin S$. We show that S^c is open. For each $x \in S$, set

$$U_x = N\left(x; \frac{|x - p|}{3}\right).$$

The family $\{U_x : x \in S\}$ is an open cover of S. Choose a finite subcover $U_{x_1}, \ldots, U_{x_n}$, and let

$$\delta = \min_{1 \leq i \leq n} \frac{|x_i - p|}{3} > 0.$$

We claim that $N(p; \delta) \cap S = \varnothing$. If instead $y \in N(p; \delta) \cap S$, then $y \in U_{x_i}$ for some i. The triangle inequality gives

$$|x_i - p| \leq |x_i - y| + |y - p| < \frac{|x_i - p|}{3} + \delta \leq \frac{2|x_i - p|}{3},$$

a contradiction. Therefore $N(p; \delta) \subseteq S^c$. Since every point of S^c is interior, S^c is open and S is closed. □

Compactness was essential in the preceding proof: it reduced infinitely many positive distances to finitely many, whose minimum remains positive. The factor $1/3$ is not important; it simply leaves enough room to separate a neighborhood of p from each selected neighborhood of x_i.

3 Compact Intervals

We next prove the difficult direction of the Heine–Borel theorem. The least upper bound property of $\mathbb{R}$ is the key ingredient.

Theorem 3.1 (Compact interval theorem). *If $a \le b$, then the closed interval $[a, b]$ is compact.*

Proof. Let $\mathcal{U}$ be an arbitrary open cover of $[a, b]$. Define

$$A = \{x \in [a, b] : [a, x] \text{ can be covered by finitely many members of } \mathcal{U}\}.$$

Some $U \in \mathcal{U}$ contains a, so $\{U\}$ covers $[a, a]$. Thus $a \in A$, while $A \subseteq [a, b]$; hence $c = \sup A$ exists.

Choose $U_0 \in \mathcal{U}$ with $c \in U_0$. Since U_0 is open, there is an $\varepsilon > 0$ such that $N(c; \varepsilon) \subseteq U_0$. Because $c = \sup A$, there exists $x \in A$ with $c - \varepsilon/2 < x \le c$; otherwise $c - \varepsilon/2$ would be an upper bound for A. Finitely many members of $\mathcal{U}$ cover $[a, x]$. Adding U_0 gives a finite cover of

$$[a, \min\{b, c + \varepsilon/2\}].$$

To verify this statement, take $y \in [a, \min\{b, c + \varepsilon/2\}]$. If $y \le x$, then the old finite collection covers y. If $y > x$, then

$$c - \varepsilon/2 < x < y \le c + \varepsilon/2,$$

so $y \in N(c; \varepsilon) \subseteq U_0$. Thus there is no gap between the part already covered and the portion covered by U_0. Consequently, $\min\{b, c + \varepsilon/2\} \in A$.

If $c < b$, then $\min\{b, c + \varepsilon/2\} > c$, contradicting that c is an upper bound for A. Hence $c = b$. The conclusion obtained above now becomes

$$b = \min\{b, b + \varepsilon/2\} \in A.$$

By the definition of A, the interval $[a, b]$ has a finite subcover. Therefore $[a, b]$ is compact. □

The proof records how far the finite-cover property has reached, takes the supremum of those locations, and uses openness to move a little farther. The only possible stopping point is the right endpoint.

4 Closed Subsets and the Heine–Borel Theorem

Theorem 4.1. *If $K \subseteq \mathbb{R}$ is compact and $F \subseteq K$ is closed, then F is compact.*

The proof is developed in Question 1 of the Compactness Worksheet. The key step is to enlarge an arbitrary open cover of F by one open set so that it covers all of K.

Theorem 4.2 (Heine–Borel). *For $S \subseteq \mathbb{R}$, the following are equivalent:*

(a) S is compact;

(b) S is closed and bounded.

The proof is completed in Question 2 of the Compactness Worksheet by combining Propositions 2.3 and 2.4, the compact interval theorem, and Theorem 4.1.

Heine–Borel is special to Euclidean spaces such as $\mathbb{R}$. The definition of compactness makes sense in general topological spaces, but "closed and bounded" is not a universal characterization.

The theorem gives a practical test in $\mathbb{R}$: first ask whether the set is bounded, and then ask whether it contains every accumulation point. Failure of either condition proves noncompactness. If both conditions hold, no separate open-cover construction is required.

Example 4.3 (Classifying subsets of $\mathbb{R}$). The intervals $[0,1]$, $[-3,7]$, and $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ are compact because they are closed and bounded. The sets $(0,1)$, $[0,1)$, and $\{1/n : n \in \mathbb{N}\}$ are bounded but not closed, so they are not compact. The sets $\mathbb{Z}$ and $[0,\infty)$ are closed but unbounded, so they are not compact.

Corollary 4.4. *A finite union of compact subsets of $\mathbb{R}$ is compact.*

Proof. Let $K_1, \ldots, K_n$ be compact, and let $\mathcal{U}$ be an arbitrary open cover of their union. For each i, the collection $\mathcal{U}$ is an open cover of K_i, so compactness supplies a finite subcollection covering K_i. The union of these n finite subcollections is finite and covers $K_1 \cup \cdots \cup K_n$. Therefore the union is compact. □

Remark 4.5. The word *finite* cannot be removed. Each singleton $\{n\}$ is compact, but $\bigcup_{n\in\mathbb{N}}\{n\} = \mathbb{N}$ is not bounded and hence is not compact.

5 First Consequences and What Comes Next

Corollary 5.1 (Extreme points of compact sets). *Every nonempty compact set $K \subseteq \mathbb{R}$ has a maximum and a minimum.*

The proof is developed in Question 3 of the Compactness Worksheet. It uses boundedness to obtain $\sup K$ and $\inf K$, and closedness to show that both values belong to K.

This corollary concerns the set itself. Later, the extreme value theorem will say that if f is continuous on a compact set K, then $f(K)$ is compact; consequently, f attains both a maximum and a minimum on K.

Our next topics reveal several important ways compactness appears in real analysis:

(a) every sequence in a compact subset of $\mathbb{R}$ has a convergent subsequence whose limit remains in the set;

(b) the continuous image of a compact set is compact;

(c) every continuous function on a compact set is uniformly continuous.

These are not separate coincidences. Compactness is the finite-control principle connecting all three results.