Continuous Functions

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MATH 312

1 Continuity at a Point

A limit describes the values of a function near a point without using its value at that point. Continuity asks that the nearby values agree, in the limit, with the actual value. The domain remains essential: we compare only inputs at which the function is defined.

Definition 1.1 (Continuity). Let $S \subseteq \mathbb{R}$, $f\colon S \to \mathbb{R}$, and $c \in S$. The function f is *continuous at* c if, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$x \in S \text{ and } |x - c| < \delta \quad \Longrightarrow \quad |f(x) - f(c)| < \epsilon.$$

It is *continuous on* S if it is continuous at every $c \in S$.

Unlike the definition of a function limit, this definition allows $x = c$. That input causes no difficulty: its output error is zero. Also, c must belong to the domain, but need not be an accumulation point of it. The choice of δ may depend on ϵ and c, but must work for every admissible x.

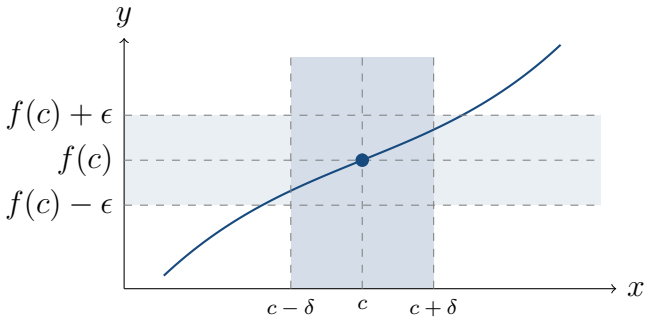


Figure 1. Continuity controls all nearby domain inputs, including c.

1.1 Accumulation Points, Endpoints, and Isolated Points

Recall $N(c;\delta) = (c-\delta, c+\delta)$ and S' denotes the set of accumulation points of S.

Proposition 1.2 (Continuity and limits). *If $c \in S \cap S'$, then f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$, with the limit taken on S. If c is an isolated point of S, every function $f\colon S \to \mathbb{R}$ is continuous at c.*

Proof. At an accumulation point, continuity implies the deleted-neighborhood limit condition. Conversely, that limit condition controls $x \neq c$, while $x = c$ has error zero. At an isolated point, choose $\delta > 0$ with $N(c;\delta) \cap S = \{c\}$. Every admissible input is then c. □

Example 1.1 (Repairing a removable discontinuity). Define $f(x) = (x^2 - 1)/(x - 1)$ for $x \neq 1$ and $f(1) = 5$. Since $\lim_{x\to 1} f(x) = 2 \neq f(1)$, the function is discontinuous at 1. Replacing $f(1)$ by 2 makes it continuous. No other assignment works, by uniqueness of limits.

Example 1.2 (The domain changes the question). The function $g(x) = \sqrt{x}$ on $[0, \infty)$ is continuous at 0: given $\epsilon > 0$, take $\delta = \epsilon^2$. Only nonnegative inputs are considered, so a left-hand limit is unnecessary.

On $S = \{0\} \cup [1, 2]$, a function with $h(0) = 100$ and $h(x) = x$ on $[1, 2]$ is continuous everywhere on S. At 0, take $\delta = 1/2$ for every $\epsilon > 0$. The gap in the domain does not create a discontinuity.

2 Sequential and Neighborhood Criteria

Theorem 2.1 (Criteria for continuity). *Let $f\colon S \to \mathbb{R}$ and $c \in S$. The following are equivalent:*

(a) f is continuous at c.

(b) For every sequence (x_n) in S with $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = f(c)$.

(c) For every neighborhood V of $f(c)$, there is a neighborhood U of c such that $f(U \cap S) \subseteq V$.

Proof. Assume (a), and let (x_n) be as in (b). Given $\epsilon > 0$, continuity supplies $\delta > 0$. Convergence supplies $N \in \mathbb{N}$ such that $|x_n - c| < \delta$ for all $n \geq N$. Thus $|f(x_n) - f(c)| < \epsilon$ for all $n \geq N$, proving (b).

To prove (b) implies (a), use the contrapositive. Failure of continuity gives a fixed $\epsilon_0 > 0$ such that for every $\delta > 0$ there exists $x \in S$ with

$$|x - c| < \delta \quad \text{and} \quad |f(x) - f(c)| \geq \epsilon_0.$$

For each $n \in \mathbb{N}$, choose such an x_n using $\delta = 1/n$. Then $\lim_{n\to\infty} x_n = c$, but $(f(x_n))$ does not converge to $f(c)$. This contradicts (b).

For (a) implies (c), choose $\epsilon > 0$ with $N(f(c);\epsilon) \subseteq V$, and let $U = N(c;\delta)$ for the corresponding continuity choice. Conversely, apply (c) to $V = N(f(c);\epsilon)$ and choose $\delta > 0$ with $N(c;\delta) \subseteq U$. This proves (a). □

Unlike the sequential criterion for a deleted-neighborhood limit, part (b) allows $x_n = c$, including constant sequences. The same proof works at endpoints and isolated points.

Example 2.1 (The Dirichlet function). Let $D(x) = 1$ for rational x and $D(x) = 0$ for irrational x. If c is rational, density of the irrationals supplies irrational x_n with $|x_n - c| < 1/n$. Then $D(x_n) = 0 \neq D(c)$. If c is irrational, use rational x_n instead. The sequential criterion shows that D is discontinuous everywhere.

3 Properties of Continuous Functions

Theorem 3.1 (Algebraic rules). *If $f, g\colon S \to \mathbb{R}$ are continuous at $c \in S$ and $a \in \mathbb{R}$, then $f+g$, af, and fg are continuous at c. If $g(c) \neq 0$, the quotient f/g is continuous at c on $T = \{x \in S : g(x) \neq 0\}$.*

Proof. For every sequence (x_n) in S converging to c, continuity gives image limits $f(c)$ and $g(c)$. The sequence limit laws give the limits of the sums, scalar multiples, and products. Apply Theorem 2.1. For the quotient, take sequences in T and use the quotient law, since $g(c) \neq 0$. □

The constant and identity functions are continuous, so these rules show that every polynomial is continuous on $\mathbb{R}$ and every rational function is continuous wherever its denominator is nonzero.

Theorem 3.2 (Composition). *Let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$, where $f(A) \subseteq B$. If f is continuous at $c \in A$ and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.*

Proof. Given $\epsilon > 0$, continuity of g gives $\eta > 0$ such that $y \in B$ and $|y - f(c)| < \eta$ imply $|g(y) - g(f(c))| < \epsilon$. Continuity of f gives $\delta > 0$ such that $x \in A$ and $|x - c| < \delta$ imply $|f(x) - f(c)| < \eta$. Since $f(x) \in B$, combining the implications proves the claim. □

Example 3.1 (Absolute value and composition). The reverse triangle inequality $\big||x| - |c|\big| \leq |x - c|$ proves continuity of absolute value with $\delta = \epsilon$. Composition with the continuous polynomial $x^2 - 1$ shows that $|x^2 - 1|$ is continuous on $\mathbb{R}$, including at $x = \pm 1$.

Proposition 3.3 (Local preservation of sign). *If f is continuous at c and $f(c) > 0$, then $f(x) > f(c)/2 > 0$ for all $x \in S$ sufficiently close to c. If $f(c) < 0$, then $f(x) < f(c)/2 < 0$ for all such x.*

Proof. Choose δ using $\epsilon = |f(c)|/2$ in the continuity definition. □

In particular, a continuous denominator that is nonzero at c stays away from zero near c. Continuity also gives a local bound: using $\epsilon = 1$, we obtain $|f(x)| < |f(c)| + 1$ near c. A local bound need not be a bound on the entire domain.

4 Compactness and Continuous Images

Continuity controls values near each individual input. Compactness turns these local controls into conclusions about the entire image. Recall that $K \subseteq \mathbb{R}$ is compact exactly when every sequence in K has a subsequence converging to a point of K.

Theorem 4.1 (Continuous image of a compact set). *If $K \subseteq \mathbb{R}$ is compact and $f\colon K \to \mathbb{R}$ is continuous, then $f(K)$ is compact.*

Proof. If $K = \varnothing$, its image is empty and compact. Otherwise, let (y_n) be any sequence in $f(K)$. Choose $x_n \in K$ with $f(x_n) = y_n$. Compactness of K supplies a subsequence (x_{n_k}) converging to some $c \in K$. By continuity,

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(c) \in f(K).$$

Thus every sequence in $f(K)$ has a subsequence converging to a point of $f(K)$. The sequential characterization makes $f(K)$ compact. □

The full input sequence need not converge: compactness supplies a convergent selection, and continuity controls the corresponding output selection. Repeated terms cause no difficulty. By Heine–Borel, $f(K)$ is closed and bounded. The converse fails: the function $f(x) = 0$ on $\mathbb{R}$ has compact image $\{0\}$ although its domain is not compact.

4.1 Where Completeness Enters

Bolzano–Weierstrass, and hence the sequential characterization used above, rests on completeness of $\mathbb{R}$. Boundedness gives a convergent subsequence; closedness keeps its limit in the domain; continuity identifies the limit of the corresponding outputs. None of these steps can replace the others.

5 Extreme Values on Compact Sets

Definition 5.1 (Absolute extrema). For $f\colon S \to \mathbb{R}$, a point $a \in S$ is an *absolute minimum point* if $f(a) \le f(x)$ for all $x \in S$. A point $b \in S$ is an *absolute maximum point* if $f(x) \le f(b)$ for all $x \in S$. The numbers $f(a)$ and $f(b)$ are the corresponding *extreme values*.

Theorem 5.2 (Extreme value theorem). *Let $K \subseteq \mathbb{R}$ be nonempty and compact, and let $f\colon K \to \mathbb{R}$ be continuous. Then there exist $a, b \in K$ such that*

$$f(a) \le f(x) \le f(b) \qquad \text{for every } x \in K.$$

Proof. Theorem 4.1 makes $f(K)$ nonempty and compact. The extreme-point property of compact subsets of $\mathbb{R}$ gives $m = \min f(K)$ and $M = \max f(K)$. Since m, M belong to the image, there are $a, b \in K$ with $f(a) = m$ and $f(b) = M$. □

To see the sequence mechanism explicitly, completeness gives $M = \sup f(K)$. For each n, choose $x_n \in K$ with

$$M - 1/n < f(x_n) \le M.$$

The squeeze theorem gives $\lim_{n\to\infty} f(x_n) = M$. Compactness supplies an input subsequence with limit $b \in K$. Its outputs still have limit M by the subsequence theorem; continuity gives limit $f(b)$, so uniqueness yields $f(b) = M$. Thus a supremum is actually attained. Apply the same argument to $-f$ for the minimum.

Example 5.1 (Extreme values need not occur at endpoints). For $f(x) = (x-1)^2$ on $K = [-1, 3]$, we have $0 \le f(x) \le 4$. The minimum value 0 occurs at $x = 1$, and the maximum value 4 occurs at both $x = -1$ and $x = 3$. The theorem guarantees existence, but neither uniqueness nor a method for locating the points.

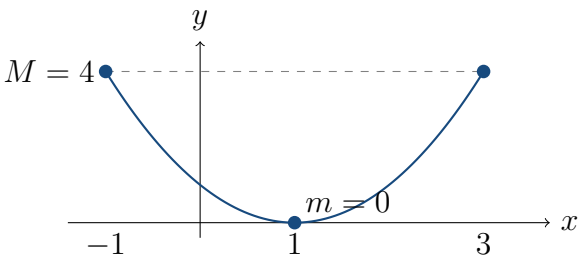


Figure 2. Extreme values are output heights; the points where they occur are inputs.

5.1 Why the Hypotheses Matter

Example 5.2 (A bounded domain missing its boundary). The identity $f(x) = x$ on $(0, 1)$ is continuous and bounded, but its image has infimum 0 and supremum 1 without attaining either. The sequence $x_n = 1 - 1/(n + 1)$ approaches the supremum, but its only possible subsequential limit is 1, outside the domain. Boundedness alone does not ensure attainment.

Example 5.3 (A compact domain without continuity). On $K = [0, 1]$, define

$$f(x) = \begin{cases} x, & 0 \le x < 1, \\ 0, & x = 1. \end{cases}$$

The image is $[0, 1)$, so its supremum 1 is not attained. Here $x_n = 1 - 1/(n + 1)$ converges to $1 \in K$, but $\lim_{n\to\infty} f(x_n) = 1 \neq f(1)$. Compactness keeps the input limit in the domain; the missing continuity prevents the output limit from being the value there.

Proposition 5.3 (A positive lower bound). *If K is nonempty and compact, $f\colon K \to \mathbb{R}$ is continuous, and $f(x) > 0$ for every $x \in K$, then there exists $m > 0$ such that $f(x) \ge m$ for every $x \in K$. Consequently $1/f$ is bounded on K.*

Proof. The extreme value theorem gives $a \in K$ with $f(a) = \min f(K)$. Put $m = f(a) > 0$. Then $0 < 1/f(x) \le 1/m$ on K. □

Pointwise positivity alone is insufficient: $f(x) = x$ on $(0, 1)$ is positive everywhere, but has no positive lower bound, and $1/f$ is unbounded.

5.2 The Domain's Extreme Points Are Different

For nonempty compact K, its smallest and largest points describe the domain. The smallest and largest points of $f(K)$ describe the extreme values of f. A minimum of f need not occur at $\min K$, as Figure 2 shows. Moreover, K need not be an interval for the extreme value theorem to apply.

6 Intermediate Values on Intervals

Theorem 6.1 (Intermediate value theorem). *Let $f\colon [a, b] \to \mathbb{R}$ be continuous, where $a < b$. If y lies strictly between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = y$.*

Proof. First suppose $f(a) < y < f(b)$. Let

$$E = \{x \in [a, b] : f(x) < y\}, \qquad c = \sup E.$$

The set E is nonempty since $a \in E$, and is bounded above by b, so completeness supplies $c \in [a, b]$. Local preservation of sign for $y - f$ at a gives $f(x) < y$ for inputs sufficiently close to a, so E contains a point greater than a and $c > a$. Local preservation of sign for $f - y$ at b gives $f(x) > y$ on a left-hand neighborhood of b; hence some number less than b is an upper bound for E, and $c < b$.

If $f(c) < y$, continuity gives $f(x) < y$ near c. Because $c < b$, this puts some $x > c$ in E, contradicting its upper bound. If $f(c) > y$, continuity gives $r > 0$ with $f(x) > y$ for domain inputs satisfying $|x - c| < r$. The supremum property supplies $x \in E$ with $c - r < x \leq c$, another contradiction. Thus $f(c) = y$.

If $f(a) > y > f(b)$, apply the first case to $-f$ and $-y$. □

Example 6.1 (A zero without an explicit formula). The polynomial $p(x) = x^3 + x - 1$ is continuous, with $p(0) = -1$ and $p(1) = 1$. The theorem gives $c \in (0, 1)$ with $p(c) = 0$. It proves existence without finding c. More generally, any continuous function with opposite signs at two endpoints has a zero between them.

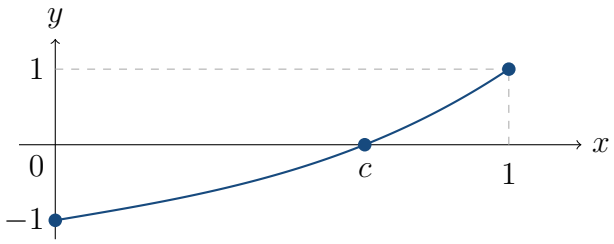


Figure 3. Continuity on the entire interval forces a crossing.

6.1 The Image of a Closed Interval

Corollary 6.2. *If $f\colon [a, b] \to \mathbb{R}$ is continuous and $a \leq b$, then*

$$f([a, b]) = [m, M], \qquad m = \min f([a, b]), \quad M = \max f([a, b]).$$

Proof. The extreme value theorem gives points $u, v \in [a, b]$ with $f(u) = m$ and $f(v) = M$. All image values lie in $[m, M]$. If $m < M$, apply the intermediate value theorem on the interval with endpoints u, v to obtain every value strictly between them. The endpoints already belong to the image. If $m = M$, the image is the singleton $\{m\}$. □

Compactness supplies the endpoints of the image; continuity on an interval supplies every value between them. On an interval that is not compact, continuity still prevents gaps in the image, but extreme values need not be attained.

Example 6.2 (Why an interval is necessary). The function $f(x) = x$ is continuous on $S = [-2, -1] \cup [1, 2]$, takes negative and positive values, and has no zero. The interval between a negative input and a positive input contains points missing from S, so the intermediate value theorem does not apply there.