Continuity and Differentiation Worksheet

Math 312: Concepts of Real Analysis

Name: _____ **Date:** _____

Directions. Use your notes on continuity, uniform continuity, derivatives, and the Mean Value Theorem. Justify each claim and identify the definitions or theorems you use. Continue on additional paper as needed.

1. **The sequential criterion for continuity.**

   Let $f : S \to \mathbb{R}$ and $c \in S$.

   (a) Suppose f is continuous at c and (x_n) is a sequence in S with $x_n \to c$. Use the definitions to prove $f(x_n) \to f(c)$.

   (b) Suppose f is discontinuous at c. Negate the definition of continuity and use $\delta = 1/n$ to construct a sequence that violates the sequential criterion.

   (c) Define $f(x) = x$ for $x < 0$ and $f(x) = 1 + x$ for $x \geq 0$. Let $x_n = (-1)^n/n$. Find the limits of the even and odd subsequences of $(f(x_n))$. What do these limits show about f at 0?

2. Uniform continuity preserves Cauchy sequences.

Let $S \subseteq \mathbb{R}$, let $f : S \to \mathbb{R}$ be uniformly continuous, and let (x_n) be a Cauchy sequence in S. We will prove that $(f(x_n))$ is Cauchy.

(a) Fix $\epsilon > 0$. Use uniform continuity to choose $\delta > 0$ and state what this choice guarantees for pairs of points in S.

(b) Use the Cauchy property of (x_n) with the δ from part (a) to choose $N \in \mathbb{N}$. State the inequality that holds for all $m, n \geq N$.

(c) For arbitrary $m, n \geq N$, combine parts (a) and (b) to bound $|f(x_n) - f(x_m)|$. Explain why this proves that $(f(x_n))$ is Cauchy.

3. Continuity without uniform continuity.

Let $f(x) = x^2 + x$ on $(0, \infty)$.

(a) Use the ϵ–δ definition to prove continuity at an arbitrary $c > 0$. Factor $f(x) - f(c)$ and bound one factor under the assumption $|x - c| < 1$.

(b) Set $x_n = n$. Choose $y_n > x_n$ such that $y_n - x_n \to 0$ while the image distances stay bounded below by a positive constant. Use these pairs to disprove uniform continuity.

(c) Prove directly that the restriction to $(1, 2)$ is uniformly continuous. Explain why the Heine–Cantor theorem cannot be applied directly on $(1, 2)$.

4. Differentiability and oscillating difference quotients.

For $x \neq 0$, let $f(x) = x\cos(1/x)$ and $g(x) = x^2\cos(1/x)$. Set $f(0) = g(0) = 0$.

(a) Prove that if F is differentiable at an interior point c of its domain, then F is continuous at c. Use an arbitrary sequence $x_n \to c$ in the domain and account for terms equal to c.

(b) Show that f is continuous at 0. For $h_n = 1/(n\pi)$, compute the difference quotient of f at 0 and find its even and odd subsequential limits. Decide whether $f'(0)$ exists.

(c) Prove $g'(0) = 0$ using the sequential criterion. Explain why the argument applies to every sequence of nonzero increments tending to zero.

5. Deriving the quotient rule.

Let $f, g : S \to \mathbb{R}$ be differentiable at an interior point c of S, and suppose $g(c) \neq 0$. Define $h(x) = f(x)/g(x)$ wherever $g(x) \neq 0$.

(a) Use continuity of g at c to show that h is defined on some open interval containing c. Explain where $g(c) \neq 0$ is needed.

(b) Before applying the product rule, show that $h'(c)$ exists. For $x \neq c$ sufficiently close to c, use $g(x)h(x) = f(x)$ to express $g(x)[h(x) - h(c)]$ in terms of $f(x) - f(c)$ and $g(x) - g(c)$. Divide by $x - c$ and $g(x)$, and apply the limit laws.

(c) Now apply the product rule to $gh = f$ at c. Solve for $h'(c)$ and express the result entirely in terms of $f(c)$, $g(c)$, $f'(c)$, and $g'(c)$.

6. Deriving the chain rule.

Let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, and $f(A) \subseteq B$. Suppose f is differentiable at an interior point c of A and g is differentiable at the interior point $d = f(c)$ of B. Define

$$Q(y) = \begin{cases} \dfrac{g(y) - g(d)}{y - d}, & y \in B,\ y \neq d, \\ g'(d), & y = d. \end{cases}$$

(a) Prove Q is continuous at d using the definition of $g'(d)$. Why is the value assigned at d useful?

(b) For $x \in A \setminus \{c\}$, express the difference quotient of $g \circ f$ as $Q(f(x))$ times the difference quotient of f. Verify the identity also when $f(x) = d$.

(c) Explain why $Q \circ f$ is continuous at c. Then take the limit as $x \to c$ in your identity to prove the chain rule.

7. Rolle's theorem and the Mean Value Theorem.

Recall that both theorems require continuity on $[a, b]$ and differentiability on (a, b), where $a < b$. Rolle's theorem also requires $f(a) = f(b)$.

(a) On $[0, 1]$, let $u(0) = 1$ and $u(x) = x$ for $0 < x \leq 1$. On $[-1, 1]$, let $v(x) = |x|$. For each function, identify the failed hypothesis and show that both theorems' conclusions fail. Compute the secant slope in each case.

(b) On $[0, 1]$, let $w(x) = x$. Show that Rolle's conclusion fails when the endpoints have unequal values, and find all $c \in (0, 1)$ that satisfy the conclusion of the Mean Value Theorem.

(c) Let I be an interval with nonempty interior. Suppose $f : I \to \mathbb{R}$ is continuous on I and differentiable on its interior, with $|f'(t)| \leq M$ there for some $M \geq 0$. Use the Mean Value Theorem to prove $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in I$ and deduce uniform continuity. Handle $M = 0$.