Darboux Sums

Thomas R. Cameron

November 5, 2025

1 Partitions

A partition P of the interval $[a, b]$ is a finite set of points $\{x_0, x_1, \dots, x_n\}$ in $[a, b]$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

If P and Q are both partitions of $[a, b]$ and $P \subseteq Q$, then Q is called a refinement of P.

For example,

$$P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$$

is a partition of the interval $[0, 1]$. Similarly,

$$Q = \left\{0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1\right\}$$

is a partition of the interval $[0, 1]$. Since $P \subseteq Q$, Q is a refinement of P.

2 Darboux Sums

Suppose that $f\colon [a, b] \to \mathbb{R}$ is bounded and let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Then, the upper Darboux sum of f with respect to P is defined by

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i,$$

and the lower Darboux sum of f with respect to P is defined by

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i,$$

where $M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i]\}$, $m_i = \inf\{f(x)\colon x \in [x_{i-1}, x_i]\}$, and $\Delta x_i = x_i - x_{i-1}$. Note that, since f is bounded, the existence of the infimum and supremum of $f(x)$ over $[x_{i-1}, x_i]$ is guaranteed by the completeness axiom.

As an example, define $f\colon [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[0, 1]$. Then, $M_i = 1$ and $m_i = 0$ for all $i \in \{1, \dots, n\}$. Therefore,

$$L(f, P) = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0$$

and

$$U(f,P) = \sum_{i=1}^{n} 1 \cdot \Delta x_i = 1.$$

As another example, define $f\colon [0,1] \to \mathbb{R}$ by $f(x) = x$. Let $n \in \mathbb{N}$ and let $P = \{0, 1/n, 2/n, \dots, 1\}$ be a partition of $[0,1]$. Then, $M_i = i/n$ and $m_i = (i-1)/n$ for all $i \in \{1, \dots, n\}$. Therefore,

$$L(f,P) = \sum_{i=1}^{n} \frac{i-1}{n} \cdot \frac{1}{n} = \frac{n^2 - n}{2n^2}$$

and

$$U(f,P) = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{n^2 + n}{2n^2}.$$

As a final example, define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 1 & \text{if } x = 0, \\ 0 & \text{if } x = 1. \end{cases}$$

Let $n \in \mathbb{N}$ and let $P = \{0, 1/n, 2/n, \dots, 1\}$ be a partition of $[0,1]$. Then, $M_1 = 1$, $M_n = 1$, and $M_i = i/n$ for all $i \in \{2, \dots, n-1\}$. Also, $m_1 = 0$, $m_n = 0$, and $m_i = (i-1)/n$ for all $i \in \{2, \dots, n-1\}$. Therefore,

$$L(f,P) = 0 \cdot \frac{1}{n} + \sum_{i=2}^{n-1} \frac{i-1}{n} \cdot \frac{1}{n} + 0 \cdot \frac{1}{n} = \frac{n^2 - 3n + 2}{2n^2}$$

and

$$U(f,P) = 1 \cdot \frac{1}{n} + \sum_{i=2}^{n-1} \frac{i}{n} \cdot \frac{1}{1} + 1 \cdot \frac{1}{n} = \frac{n^2 + n - 2}{2n^2} + \frac{1}{n}.$$

The following theorem provides bounds on the upper and lower Darboux sums for partitions and their refinements.

Theorem 2.1. *Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Let P and Q are partitions of $[a,b]$ such that Q is a refinement of P. Then,*

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$

Proof. Let x^* be a point in $[a,b]$ that is not in P. Then, there is a $k \in \{1, \dots, n\}$ such that $x_{k-1} < x^* < x_k$. Define

$$\begin{aligned} t_1 &= \inf\{f(x) \colon x \in [x_{k-1}, x^*]\}, \\ t_2 &= \inf\{f(x) \colon x \in [x^*, x_k]\}, \\ s_1 &= \sup\{f(x) \colon x \in [x_{k-1}, x^*]\}, \\ s_2 &= \sup\{f(x) \colon x \in [x^*, x_k]\}. \end{aligned}$$

Then, $t_1 \ge m_k$, $t_2 \ge m_k$, $s_1 \le M_k$, and $s_2 \le M_k$.

Now, let $P^* = P \cup \{x^*\}$. Then, the terms in $L(f,P^*)$ and $L(f,P)$ are all the same except those over the subinterval $[x_{k-1}, x_k]$. Thus, we have

$$\begin{aligned} L(f,P^*) - L(f,P) &= t_1(x^* - x_{k-1}) + t_2(x_k - x^*) - m_k(x_k - x_{k-1}) \\ &= (t_1 - m_k)(x^* - x_{k-1}) + (t_2 - m_k)(x_k - x^*) \ge 0. \end{aligned}$$

Similarly, the terms in $U(f,P^*)$ and $U(f,P)$ are all the same except those over the subinterval $[x_{k-1}, x_k]$. Thus, we have

$$\begin{aligned} U(f,P^*) - U(f,P) &= s_1(x^* - x_{k-1}) + s_2(x_k - x^*) - M_k(x_k - x_{k-1}) \\ &= (s_1 - M_k)(x^* - x_{k-1}) + (s_2 - M_k)(x_k - x^*) \le 0. \end{aligned}$$

Therefore, adding points to a partition can only increase the lower Darboux sum and can only decrease the upper Darboux sum. By repeating this argument to every point in Q that is not in P, we conclude that

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

□

The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let* $f \colon [a, b] \to \mathbb{R}$ *be bounded. Let* P *and* Q *be partitions of* $[a, b]$*. Then,* $L(f, P) \leq U(f, Q)$.

Proof. Since $P \cup Q$ is a refinement of both P and Q, Theorem 2.1 implies that

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q).$$

□