The Derivative

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1 The Derivative

Let $S \subseteq \mathbb{R}$ and let $f\colon S \to \mathbb{R}$. The *derivative* of f at an interior point $c \in S$ is defined by

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c},$$

provided the limit exists and is finite. In this case, we say that f is *differentiable* at c.

As an example, consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. Then, for any $c \in \mathbb{R}$ we have

$$f'(c) = \lim_{x\to c} \frac{x - c}{x - c} = \lim_{x\to c}(1) = 1.$$

As another example, consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then, for any $c \in \mathbb{R}$ we have

$$f'(c) = \lim_{x\to c} \frac{x^2 - c^2}{x - c} = \lim_{x\to c}(x + c) = 2c.$$

The sequential criterion for limits can be applied to the limit definition of the derivative. As a result, we have the following theorem.

Theorem 1.1. *Let $S \subseteq \mathbb{R}$, $f\colon S \to \mathbb{R}$, and c be an interior point of S. Then, f is differentiable at c if and only if for every $x\colon \mathbb{N} \to S \setminus \{c\}$ the sequence*

$$\frac{f(x_n) - f(c)}{x_n - c}$$

converges.

Theorem 1.1 is particularly useful for showing when a function is not differentiable. For example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$. Then, f is not differentiable at $c = 0$. Indeed, let $x_n = \frac{(-1)^n}{n}$ which converges to $c = 0$ However, the sequence

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{1/n - 0}{(-1)^n/n - 0} = (-1)^n$$

does not converge. The previous example demonstrates that a function can be continuous at a point but not differentiable at that point. On the other hand, the following result shows that differentiability at a point implies continuity at that point.

Theorem 1.2. *Let $S \subseteq \mathbb{R}$, $f\colon S \to \mathbb{R}$, and c be an interior point of S. If f is differentiable at c, then f is continuous at c*

Proof. Suppose that f is differentiable at c. Since c is an interior point of S it is an accumulation point of S. Thus, we can show that f is continuous at c by showing that $\lim_{x\to c} f(x) = f(c)$. To that end, let $x\colon \mathbb{N} \to S \setminus \{c\}$ be a sequence that converges to c. Then, the sequence

$$\frac{f(x_n) - f(c)}{x_n - c}$$

converges and we denote its limiting value by $f'(c)$. Now,

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty}\left((x_n - c)\frac{f(x_n)-f(c)}{x_n - c} + f(c)\right)\\ &= \lim_{n\to\infty}(x_n - c)\cdot\lim_{n\to\infty}\frac{f(x_n)-f(c)}{x_n - c} + f(c)\\ &= 0\cdot f'(c) + f(c) = f(c).\end{aligned}$$

□

2 Derivative Rules

The following theorem presents useful rules for taking the derivative of sums, products, and quotients.

Theorem 2.1. *Let $S \subseteq \mathbb{R}$, c be an interior point of S, $f\colon S \to \mathbb{R}$, and $g\colon S \to \mathbb{R}$. If f and g are differentiable at c, then the following properties hold.*

(a) $f + g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$.

(b) $f \cdot g$ is differentiable at c and $(f\cdot g)'(c) = f'(c)\cdot g(c) + f(c)\cdot g'(c)$

(c) If $g(c) \neq 0$, then f/g is differentiable at c and $(f/g)'(c) = \frac{f'(c)\cdot g(c) - f(c)\cdot g'(c)}{g(c)^2}$

As a corollary of Theorem 2.1, we will prove the constant multiple rule.

Corollary 2.2. *Let $S \subseteq \mathbb{R}$, c be an interior point of S, $f\colon S \to \mathbb{R}$, and $k \in \mathbb{R}$. If f is differentiable at c, then $k\cdot f$ is differentiable at c and $(k\cdot f)'(c) = k\cdot f'(c)$.*

Proof. Note that a consant function has a derivative equal to zero everywhere. Hence, Theorem 2.1 (b) implies that $k \cdot f$ is differentiable at c and

$$(k\cdot f)'(c) = 0\cdot f(c) + k\cdot f'(c) = k\cdot f'(c).$$

□

Also, as a corollary of Theorem 2.1, we will prove the power rule (for natural number powers).

Corollary 2.3. *Let $f\colon \mathbb{R}\to\mathbb{R}$ be defined by $f(x) = x^n$, where $n \in \mathbb{N}$. Then f is differentiable at each $c \in \mathbb{R}$ and $f'(c) = nc^{n-1}$.*

Proof. When $n = 1$ the result holds since the derivative of $f(x) = x$ is equal to one everywhere. Let $n \in \mathbb{N}$ and suppose that for any $c \in \mathbb{R}$, $f(x) = x^n$ is differentiable at c and $f'(c) = nc^{n-1}$. By the product rule, $x \cdot f(x) = x^{n+1}$ is differentiable at c and

$$(x\cdot f(x))'(c) = 1\cdot f(c) + c\cdot f'(c) = c^n + nc^n = (n+1)c^n.$$

Therefore, the result holds for all $n \in \mathbb{N}$ by the principal of mathematical induction. □

The following theorem presents the chain rule.

Theorem 2.4. *Let $f\colon A \to \mathbb{R}$ and $g\colon B\to\mathbb{R}$ such that $f(A) \subseteq B$. Suppose that c is an interior point of A and $f(c)$ is an interior point of B. If f is differentiable at c and g is differentiable at $f(c)$, then $g\circ f$ is differentiable at c and*

$$(g\circ f)'(c) = g'(f(c))f'(c).$$