The Derivative Worksheet 2

Thomas R. Cameron

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1 Exercises

I. Let $f\colon A \to \mathbb{R}$, $g\colon B \to \mathbb{R}$, and $f(A) \subseteq B$. Suppose that f is differentiable at $c \in A$ and g is differentiable at $f(c)$. Then, $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Proof. Let $x\colon \mathbb{N} \to A \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$. If f is constant in a neighborhood of c, then there is a $\delta \in \mathbb{R}_{>0}$ such that $f(x) = f(c)$ for all $x \in N(c;\delta) \cap A$. Since $\lim_{n\to\infty} x_n = c$, there is a $N \in \mathbb{N}$ such that $f(x_n) = f(c)$ for all $n \geq N$. Therefore,

$$(g \circ f)'(c) = \lim_{n\to\infty} \frac{g(f(x_n)) - g(f(c))}{x_n - c} = 0.$$

Moreover, since f is constant in a neighborhood of c, $f'(c) = 0$. Thus, $(g \circ f)'(c) = g'(f(c))f'(c) = 0$.

Suppose that f is not constant in any neighborhood of c. Then, there is a $\delta \in \mathbb{R}_{>0}$ such that $f(x) \neq f(c)$ for all $x \in N(c;\delta) \cap A$. Since $\lim_{n\to\infty} x_n = c$, there is a $N \in \mathbb{N}$ such that $f(x_n) \neq f(c)$ for all $n \geq N$. Therefore,

$$\begin{aligned}
(g \circ f)'(c) &= \lim_{n\to\infty} \frac{g(f(x_n)) - g(f(c))}{x_n - c} \\
&= \lim_{n\to\infty} \frac{g(f(x_n)) - g(f(c))}{f(x_n) - f(c)} \frac{f(x_n) - f(c)}{x_n - c} \\
&= \lim_{n\to\infty} \frac{g(f(x_n)) - g(f(c))}{f(x_n) - f(c)} \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} \\
&= g'(f(c))f'(c)
\end{aligned}$$

$\square$

II. Prove the Cauchy Mean Value Theorem: Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then, there is a $c \in (a,b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

Proof.

- Define $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$.
- Apply Rolle's theorem to $h(x)$.

$\square$

III. Prove L'Hopitals Rule: Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose there is a $c \in (a,b)$ such that $f(c) = g(c) = 0$ and $g'(x) \neq 0$ for all x in a deleted neighborhood of c. If $\lim_{x\to c} f'(x)/g'(x) = L$, for some $L \in \mathbb{R}$, then $\lim_{x\to c} f(x)/g(x) = L$.

Proof.

- Let $x\colon \mathbb{N} \to [a,b] \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$.
- For each n, use the Cauchy Mean Value Theorem to define c_n between x_n and c such that

$$[f(x_n) - f(c)]g'(c_n) = [g(x_n) - g(c)]f'(c_n).$$

- Let $\delta \in \mathbb{R}_{>0}$ such that $g'(x) \neq 0$ for all $x \in N^*(c;\delta) \cap (a,b)$. Use Rolle's theorem to conclude that there is a $N \in \mathbb{N}$ such that $g(x_n) \neq 0$ for all $n \geq N$.
- Conclude that

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(c)}{g(x_n) - g(c)} = \frac{f'(c_n)}{g'(c_n)},$$

for all $n \geq N$. Therefore, if $\lim_{x\to c} f'(x)/g'(x) = L$, for some $L \in \mathbb{R}$, then $\lim_{x\to c} f(x)/g(x) = L$.

□

IV. Prove Taylor's Theorem: Let f and its first n derivatives be continuous on $[a,b]$ and differentiable on (a,b), and let $x_0 \in [a,b]$. Then, for each $x \in [a,b]$ with $x \neq x_0$ there exists a c between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Proof.

- Fix $x \in [a,b]$ with $x \neq x_0$. Let M be the unique solution of

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + M(x - x_0)^{n+1}.$$

- Define

$$F(t) = f(t) + f'(t)(x - t) + \cdots \frac{f^{(n)}(t)}{n!}(x - t)^n + M(x - t)^{n+1}$$

- Apply Rolle's Theorem to $F(t)$.

□