The Derivative

Thomas R. Cameron

MATH 312

1 The Derivative at a Point

Continuity asks whether nearby inputs have nearby outputs. Differentiability asks a more precise question: does the change in output have a limiting ratio to the change in input? This ratio describes a local rate of change and the slope of a tangent line.

Definition 1.1 (Derivative). Let $S \subseteq \mathbb{R}$, $f\colon S \to \mathbb{R}$, and let c be an interior point of S. The function f is *differentiable at* c if the following limit exists as a finite real number:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

The number $f'(c)$ is the *derivative of* f *at* c. If f is differentiable at every point of an open set $U \subseteq S$, its *derivative function* is $f'\colon U \to \mathbb{R}$.

The quotients are defined only for $x \neq c$ or $h \neq 0$. The two limits agree by the substitution $h = x - c$. Because c is interior, all sufficiently small positive and negative increments are allowed. Throughout these notes, differentiability means this two-sided, finite derivative; endpoint derivatives require a separate one-sided convention.

Example 1.1 (Constants, the identity, and a quadratic). For $f(x) = k$, the difference quotient is zero, so $f'(c) = 0$. For $f(x) = x$, it is one, so $f'(c) = 1$. For $f(x) = x^2$,

$$\frac{f(x) - f(c)}{x - c} = \frac{(x-c)(x+c)}{x - c} = x + c \quad (x \neq c),$$

and hence $f'(c) = 2c$. The cancellation is valid on the deleted neighborhood used in the limit.

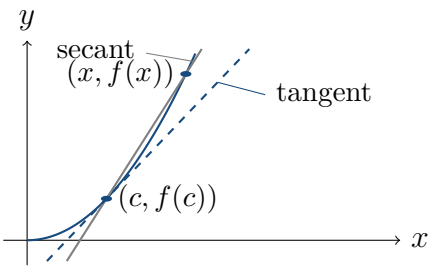


Figure 1. Secant slopes approach the tangent slope as $x \to c$.

2 Sequential Criteria and Failure of Differentiability

The sequential criterion for function limits applies to the difference quotient. The input sequence must approach the point where the derivative is being tested.

Theorem 2.1 (Sequential criterion). *Let $f\colon S \to \mathbb{R}$, let c be an interior point of S, and let $L \in \mathbb{R}$. Then $f'(c) = L$ if and only if, for every sequence (x_n) in $S \setminus \{c\}$ with $\lim_{n\to\infty} x_n = c$,*

$$\lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} = L.$$

Proof. Apply the sequential criterion for limits to $q(x) = (f(x) - f(c))/(x - c)$ on $S \setminus \{c\}$. Since c is interior to S, it is an accumulation point of this domain. □

To disprove differentiability, it suffices to find one approaching sequence whose quotients fail to converge to a finite number, or two approaching sequences whose quotient limits differ.

Example 2.1 (A corner). The function $f(x) = |x|$ is continuous at zero, but

$$\frac{|x| - |0|}{x - 0} = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

Taking $x_n = 1/n$ and $y_n = -1/n$ gives different quotient limits. Alternatively, $z_n = (-1)^n/n$ gives the nonconvergent quotient sequence $(-1)^n$. Thus $f'(0)$ does not exist.

Example 2.2 (An infinite slope is not a derivative). For $g(x) = x^{1/3}$, the quotient at zero is $x^{-2/3} = 1/|x|^{2/3}$. Along $x_n = 1/n^3$ the quotients equal n^2, so they have no finite limit. The graph has a vertical tangent at zero, but g is not differentiable there under our definition.

Example 2.3 (Oscillation of the difference quotient). Define $p(0) = 0$ and $p(x) = x\sin(1/x)$ for $x \neq 0$. Since $|p(x)| \leq |x|$, the squeeze theorem proves continuity at zero. Its difference quotient is $\sin(1/x)$. For

$$x_n = \frac{1}{2\pi n + \pi/2}, \qquad y_n = \frac{1}{2\pi n + 3\pi/2},$$

both input sequences tend to zero, while the quotients are respectively 1 and -1. Thus p is not differentiable at zero.

The corner, infinite slope, and oscillation illustrate three different failures of the same finite-limit requirement. Continuity alone rules out none of them.

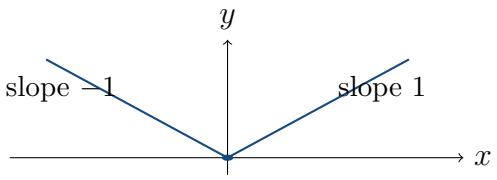


Figure 2. Continuity at a corner does not give a limiting secant slope.

3 Continuity and Local Linear Approximation

Theorem 3.1 (Differentiability implies continuity). *If $f\colon S \to \mathbb{R}$ is differentiable at an interior point c of S, then f is continuous at c.*

Proof. For $x \neq c$, write

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c).$$

The quotient tends to the finite number $f'(c)$ and $x - c$ tends to zero. The product law gives $\lim_{x\to c}(f(x) - f(c)) = 0$, or $\lim_{x\to c} f(x) = f(c)$. This is continuity at c. □

The converse fails, as $|x|$ shows. The contrapositive is useful: a function discontinuous at c cannot be differentiable there.

Proposition 3.2 (A precise linear approximation). *A function f is differentiable at an interior point c with $f'(c) = L$ if and only if, for all sufficiently small h,*

$$f(c + h) = f(c) + Lh + h\rho(h), \qquad \lim_{h\to 0} \rho(h) = 0,$$

where we may set $\rho(0) = 0$.

Proof. If $f'(c) = L$, define $\rho(h) = (f(c + h) - f(c))/h - L$ for $h \neq 0$ and $\rho(0) = 0$. The derivative limit gives $\rho(h) \to 0$ and the stated identity. Conversely, dividing the identity by $h \neq 0$ gives a difference quotient equal to $L + \rho(h)$, whose limit is L. □

The tangent line is $y = f(c) + f'(c)(x - c)$. Its error is small *relative to the input change*: for each $\epsilon > 0$, there is $\delta > 0$ such that

$$0 < |h| < \delta \quad \Longrightarrow \quad |f(c + h) - f(c) - f'(c)h| < \epsilon|h|.$$

Requiring only that the error tend to zero would be equivalent to continuity at c for any fixed L. Differentiability requires the error divided by $|h|$ to tend to zero; the slope L satisfying this stronger condition is unique.

Example 3.1 (An exact remainder). For $f(x) = x^2$ at $c = 1$, the identity $(1 + h)^2 = 1 + 2h + h^2$ gives the tangent $y = 1 + 2(x - 1)$ and remainder $h^2 = h\rho(h)$ with $\rho(h) = h$. Thus the error divided by $|h|$ is $|h| \to 0$.

Example 3.2 (A derivative need not be continuous). Let $q(0) = 0$ and $q(x) = x^2 \sin(1/x)$ for $x \neq 0$. At zero its quotient is $x \sin(1/x) \to 0$, so $q'(0) = 0$. The rules proved below and the usual trigonometric derivatives give

$$q'(x) = 2x \sin(1/x) - \cos(1/x) \qquad (x \neq 0).$$

At $x_n = 1/(2\pi n)$, $q'(x_n) = -1$. Hence q' is not continuous at zero, even though q is differentiable everywhere.

Remark 3.3 (Uniqueness of the tangent slope). If both L and M satisfy the approximation condition, the difference quotient tends to both L and M. Uniqueness of limits gives $L = M$. Thus the linear approximation has a unique slope, including when that slope is zero.

4 Algebraic Rules for Derivatives

Theorem 4.1 (Sum, product, and quotient rules). *Let $f, g\colon S \to \mathbb{R}$ be differentiable at an interior point c of S. Then $f+g$ and fg are differentiable at c, with*

$$(f+g)'(c) = f'(c) + g'(c), \qquad (fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

If $g(c) \neq 0$, then f/g is differentiable at c on its domain $T = \{x \in S : g(x) \neq 0\}$, with

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

Proof. For the sum, for $x \neq c$ we have

$$\frac{(f+g)(x) - (f+g)(c)}{x-c} = \frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c}.$$

The two terms tend to $f'(c)$ and $g'(c)$, respectively, so the sum law gives the claimed derivative.

For the product, add and subtract $f(c)g(x)$ in the numerator:

$$\frac{f(x)g(x) - f(c)g(c)}{x-c} = \frac{f(x)-f(c)}{x-c}g(x) + f(c)\frac{g(x)-g(c)}{x-c}.$$

By Theorem 3.1, $g(x) \to g(c)$. The limit laws now give the product formula.

For the quotient, continuity of g and $g(c) \neq 0$ give a neighborhood of c where $|g(x)| > |g(c)|/2$. In particular, c is interior to T. For nearby $x \neq c$,

$$\frac{f(x)/g(x) - f(c)/g(c)}{x-c} = \frac{g(c)\dfrac{f(x)-f(c)}{x-c} - f(c)\dfrac{g(x)-g(c)}{x-c}}{g(x)g(c)}.$$

The numerator tends to $g(c)f'(c) - f(c)g'(c)$ and the denominator tends to $g(c)^2 \neq 0$. The quotient limit law completes the proof. □

Corollary 4.2 (Constant multiples and differences). *If f, g are differentiable at c and $a, b \in \mathbb{R}$, then*

$$(af + bg)'(c) = af'(c) + bg'(c).$$

Proof. The derivative of the constant function a is zero. The product rule gives $(af)'(c) = 0 \cdot f(c) + af'(c)$. Apply the sum rule to $af + bg$. Taking $b = -1$ includes subtraction. □

Example 4.1 (A reciprocal directly from the definition). For $r(x) = 1/x$ and $c \neq 0$,

$$\frac{1/x - 1/c}{x-c} = -\frac{1}{xc} \longrightarrow -\frac{1}{c^2}.$$

This agrees with the quotient rule applied to the constant and identity functions. At $c = 0$ there is no derivative question because $r(0)$ is undefined.

5 Powers and Worked Applications

Corollary 5.1 (Natural number powers). *For $n \in \mathbb{N}$, the function $f(x) = x^n$ is differentiable on $\mathbb{R}$, and $f'(c) = nc^{n-1}$. When $n = 1$, the derivative is 1 at every c, including zero.*

Proof. For $n = 1$, this is the derivative of the identity. Suppose the assertion holds for $n \in \mathbb{N}$. Applying the product rule to $x^{n+1} = x \cdot x^n$ gives

$$\frac{d}{dx}x^{n+1} = 1 \cdot x^n + x \cdot nx^{n-1} = (n+1)x^n.$$

Mathematical induction proves the claim for all $n \in \mathbb{N}$. □

Together with linearity, this proves that every polynomial is differentiable on $\mathbb{R}$. The quotient rule then proves that every rational function is differentiable wherever its denominator is nonzero. For $m \in \mathbb{N}$, applying it to $1/x^m$ also gives $(x^{-m})' = -mx^{-m-1}$ for $x \neq 0$.

Example 5.1 (Combining algebraic rules). For $F(x) = (x^2 - 1)/(x^2 + 1)$, the denominator is positive everywhere, and

$$F'(x) = \frac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

Thus $F(1) = 0$ and $F'(1) = 1$, so the tangent at $x = 1$ is $y = x - 1$.

Example 5.2 (Matching pieces at a joining point). Let $f(x) = x^2$ for $x \leq 1$ and $f(x) = ax + b$ for $x > 1$. Continuity at 1 requires $a + b = 1$. Under this condition, the difference quotient at 1 is

$$\frac{f(x) - f(1)}{x - 1} = \begin{cases} x + 1, & x < 1, \\ a, & x > 1. \end{cases}$$

The one-sided limits agree precisely when $a = 2$, giving $b = -1$. Hence the unique differentiable joining is the line $2x - 1$, tangent to the parabola at $(1, 1)$.

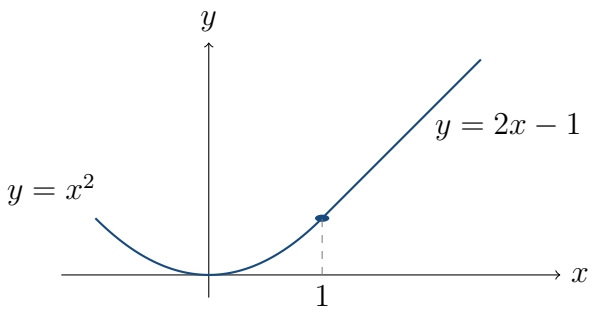


Figure 3. Matching values gives continuity; matching slopes gives differentiability.

Example 5.3 (Simplification does not fill a domain gap). The function $r(x) = (x^2 - 1)/(x - 1)$ equals $x + 1$ only for $x \neq 1$. Thus $r'(x) = 1$ on its domain, but $r'(1)$ is undefined. Defining $r(1) = 2$ gives a new function, differentiable at 1 with derivative 1. Any other assignment is discontinuous there and therefore cannot be differentiable.

6 The Chain Rule

Theorem 6.1 (Chain rule). *Let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$, with $f(A) \subseteq B$. Suppose c is interior to A and $f(c)$ is interior to B. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c and*

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Proof. Put $b = f(c)$ and extend the difference quotient of g by defining

$$G(y) = \begin{cases} \dfrac{g(y) - g(b)}{y - b}, & y \in B \setminus \{b\}, \\ g'(b), & y = b. \end{cases}$$

Differentiability of g at b says $\lim_{y \to b} G(y) = g'(b) = G(b)$, so G is continuous at b. For *every* $y \in B$, including $y = b$,

$$g(y) - g(b) = G(y)(y - b).$$

Substitute $y = f(x)$ and divide by $x - c$ for $x \neq c$:

$$\frac{g(f(x)) - g(f(c))}{x - c} = G(f(x))\frac{f(x) - f(c)}{x - c}.$$

Differentiability makes f continuous at c, so continuity of composition gives $G(f(x)) \to G(b) = g'(b)$. The second factor tends to $f'(c)$, proving the formula. □

We cannot simply divide by $f(x) - f(c)$: it can vanish at points arbitrarily close to c, even if f is not constant on any neighborhood. The extended quotient makes the identity valid at all such inputs.

Example 6.1 (Repeated inner values). Let $f(0) = 0$ and $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, and let $g(y) = y^2$. We proved $f'(0) = 0$, and $f(1/(\pi n)) = f(0)$ for every $n \in \mathbb{N}$. The extended quotient for g at zero is $G(y) = y$ for all y, including zero. Thus the proof applies, and $(g \circ f)'(0) = 2f(0)f'(0) = 0$.

Example 6.2 (Composition and a zero inner derivative). For $H(x) = (1 + x^2)^3$, the power and chain rules give

$$H'(x) = 3(1 + x^2)^2(2x) = 6x(1 + x^2)^2.$$

In particular, $H'(0) = 0$ although the outer derivative at 1 equals 3. The inner derivative measures how fast the input to the outer function changes.

Differentiability is a local property, while uniform continuity concerns all pairs of inputs in a domain. For instance, x^2 is differentiable everywhere but is not uniformly continuous on $\mathbb{R}$: the inputs n and $n + 1/n$ approach one another, while their output difference is $2 + 1/n^2$.