Exam II Worksheet

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Exercises

I. Classify each of the following sets as open, closed, neither, or both. Justify your answer.

(a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

(b) $\mathbb{N}$

(c) $\mathbb{Q}$

(d) $\bigcap_{n=1}^{\infty}(0, 1/n)$

(e) $\left\{x \colon |x-5| \leq \frac{1}{2}\right\}$

(f) $\left\{x \colon x^2 > 0\right\}$

II. Find the closure of each set in I.

III. Let $S \subseteq \mathbb{R}$ and let S' denote the accumulation points of S. Prove that S' is closed.

IV. Show that each of the following sets is not compact by describing an open cover that has no finite subcover.

(a) $[1, 3)$

(b) $\mathbb{N}$

(c) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

V. Let $S \subseteq \mathbb{R}$ be closed and bounded. Show that every open cover of S has a finite subcover; hint: see Theorem 1.2 from the class notes.

VI. Prove the Bolzano-Weierstrass theorem: If $S \subseteq \mathbb{R}$ is infinite and bounded, then S has an accumulation point; hint: don't use results that depend on the Bolzano-Weierstrass theorem.

VII. Find a sequence of real numbers satisfying each set of properties.

(a) Cauchy, but not monotone.

(b) Monotone, but not Cauchy.

(c) Bounded, bot not Cauchy.

VIII. Let $f \colon [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a) < 0 < f(b)$. Prove that there exists a $c \in (a, b)$ such that $f(c) = 0$.

IX. Let $f \colon [a, b] \to \mathbb{R}$ be continuous. Prove that for each y between $f(a)$ and $f(b)$ there is a $c \in (a, b)$ such that $f(c) = y$.

X. Prove that $f \colon (a, b) \to \mathbb{R}$ is uniformly continuous if and only if f can be extended continuous to $\hat{f} \colon [a, b] \to \mathbb{R}$.