Exam III Worksheet Solutions

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Exercises

I. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Note that $x = 0$ is an accumulation point of $\mathbb{R}$. Furthermore, for all $x \neq 0$,

$$-x \leq x\sin\left(\frac{1}{x}\right) \leq x.$$

Therefore, $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0 = f(0)$, which implies that f is continuous at $x = 0$.

However, we will show that $f(x)$ is not differentiable at $x = 0$. To this end, let $s_n = \frac{2}{n\pi}$, for each $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} s_n = 0$. However, the following sequence

$$\frac{f(s_n) - f(0)}{s_n - 0} = \frac{s_n \sin\left(\frac{1}{s_n}\right)}{s_n} = \sin\left(\frac{1}{s_n}\right) = \sin\left(\frac{n\pi}{2}\right) \tag{1}$$

is not convergent. Indeed, let $\epsilon = 1/2$. Then, for any $N \in \mathbb{N}$, let $n \geq N$ be even and $m \geq N$ be odd, so that

$$\left|\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{m\pi}{2}\right)\right| = 1 > \epsilon.$$

Therefore, the sequence in (1) is not Cauchy and thus not convergent. Since this sequence is not convergent, the following limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist. Hence, f is not differentiable at $x = 0$.

II. Let $f\colon [a, b] \to \mathbb{R}$ be differentiable at $c \in (a, b)$. Then, f is continuous at c.

Proof. Since $c \in (a, b)$ is an accumulation point, we can show that f is continuous at c by establishing $\lim_{x\to c} f(x) = f(c)$. To that end, let $s\colon \mathbb{N} \to \mathbb{R} \setminus \{c\}$ be a sequence that converges to c. Since f is differentiable at c, the following sequence

$$\frac{f(s_n) - f(c)}{s_n - c}$$

converges and we denote its limiting value by $f'(c)$. Now,

$$\begin{aligned} \lim_{n\to\infty} f(s_n) &= \lim_{n\to\infty} \left((s_n - c)\frac{f(s_n) - f(c)}{s_n - c} + f(c) \right) \\ &= \lim_{n\to\infty} (s_n - c) \lim_{n\to\infty} \frac{f(s_n) - f(c)}{s_n - c} + f(c) \\ &= 0 \cdot f'(c) + f(c) = f(c) \end{aligned}$$

□

VI. Let $f(x) = x^3$ on $[0, 1]$.

(a) Let $P = \{0, 1/n, 2/n, \dots, 1\}$ be a partition of $[0, 1]$ for each $n \in \mathbb{N}$.

(b) Since $f(x) = x^3$ is an increasing function, the upper Darboux sum is given by

$$U(f, P) = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n} = \frac{1}{n^4} \sum_{i=1}^{n} i^3.$$

Similarly, the lower Darboux sum is given by

$$L(f, P) = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^3 \frac{1}{n} = \frac{1}{n^4} \sum_{i=1}^{n-1} i^3.$$

The sum of cubes can be written as $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$. Therefore, the upper Darboux sum is

$$U(f, P) = \frac{(n+1)^2}{4}$$

and the lower Darboux sum is

$$L(f, P) = \frac{(n-1)^2}{4}.$$

So, $\lim_{n\to\infty} L(f, P) = \lim_{n\to\infty} U(f, P) = 1/4$.

(c) Since $L(f)$ is the supremum of the set of lower Darboux sums over all partitions of $[0, 1]$, it follows that $L(f) \geq 1/4$. Indeed, if $L(f) < 1/4$, then there exists a $n \in \mathbb{N}$ such that $L(f, P) > L(f)$, which is a contradiction to $L(f)$ being an upper bound. Similarly, $U(f) \leq 1/4$. Since $L(f) \geq U(f)$, it follows that

$$\frac{1}{4} \leq L(f) \leq U(f) \leq \frac{1}{4}.$$

Therefore, $L(f) = U(f) = 1/4$

VII. Let $f\colon [a, b] \to \mathbb{R}$ be bounded.

(a) f is Riemann integrable if $L(f) = U(f)$.

(b) f is Riemann integrable if and only if for all $\epsilon \in \mathbb{R}_{>0}$ there exists a partition P such that $U(f, P) - L(f, P) < \epsilon$.

(c) If f is monotone, then f is Riemann integrable.

Proof. Suppose that f is increasing. Then, $f(a) \leq f(x) \leq f(b)$ for all $x \in [a, b]$; hence, f is bounded on $[a, b]$. Now, let $\epsilon \in \mathbb{R}_{>0}$. Then, there is a $\delta \in \mathbb{R}_{>0}$ such that

$$\delta[f(b) - f(a)] < \epsilon.$$

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ such that $\Delta x_i \leq \delta$ for all $i \in \{1, \dots, n\}$. Since f is increasing, it follows that

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\Delta x_i \\ &\leq \delta \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \\ &= \delta[f(b) - f(a)] < \epsilon. \end{aligned}$$

A similar argument can be made when f is decreasing. □

(d) If f is continuous, then f is Riemann integrable.

Proof. Since f is continuous on $[a,b]$, which is compact, it follows that f is uniformly continuous. Let $\epsilon \in \mathbb{R}_{>0}$. Then, there is a $\delta \in \mathbb{R}_{>0}$ such that for all $x, y \in [a,b]$,

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

Let P be any partition of $[a,b]$ such that $\Delta x_i \leq \delta$. Since f is continuous, for each $i \in \{1,\ldots,n\}$, f attains its maximum and minimum value on $[x_{i-1}, x_i]$. That is, there exists $s_i, t_i \in [x_{i-1}, x_i]$ such that $m_i = f(s_i)$ and $M_i = f(t_i)$. Since $|s_i - t_i| < \delta$, it follows that $[M_i - m_i] < \frac{\epsilon}{b-a}$. Therefore, we have

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} [M_i - m_i]\Delta x_i \\ &< \frac{\epsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \epsilon. \end{aligned}$$

□

IX. Let f be Riemann integrable.

(a) Suppose that $F\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then, if $F'(x) = f(x)$ for all $x \in (a,b)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Let P be any partition of $[a,b]$. For each $i \in \{1,2,\ldots,n\}$, the mean value theorem states that there is a $t_i \in (x_{i-1}, x_i)$ such that

$$F'(t_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}},$$

that is, $f(t_i)\Delta x_i = F(x_i) - F(x_{i-1})$. Since $\sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n}[F(x_i) - F(x_{i-1})] = F(b) - F(a)$,

$$L(f,P) \leq F(b) - F(a) \leq U(f,P).$$

Since the above bound holds for all partitions P of $[a,b]$,

$$L(f) \leq F(b) - F(a) \leq U(f).$$

Therefore, since f is Riemann integrable,

$$\int_a^b f(x)dx = L(f) = U(f) = F(b) - F(a).$$

□

(b) Define

$$F(x) = \int_a^x f(t)dt,$$

for each $x \in [a,b]$. Then, $F\colon [a,b] \to \mathbb{R}$ is uniformmly continuous on $[a,b]$. Moreover, if f is continuous at c, then F is differentiable at c and $F'(c) = f(c)$.

Proof. Since f is Riemann integrable, it is bounded. So, there exists an $M \in \mathbb{R}_{>0}$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Let $\epsilon \in \mathbb{R}_{>0}$ and $\delta = \epsilon/M$. Also, note that, if $x \leq y$,

$$\begin{aligned} |F(x) - F(y)| &= \left| \int_a^x f(t)dt - \int_a^y f(t)dt \right| \\ &= \left| - \int_x^y f(t)dt \right| \\ &\leq \int_x^y |f(t)|\, dt \leq M\,|x - y| \end{aligned}$$

Also, if $x > y$,

$$\begin{aligned} |F(x) - F(y)| &= \left| \int_a^x f(t)dt - \int_a^y f(t)dt \right| \\ &= \left| \int_y^x f(t)dt \right| \\ &\leq \int_y^x |f(t)dt| \leq M\,|x - y| \end{aligned}$$

Hence, for all $x, y \in [a, b]$, if $|x - y| < \delta$ the

$$|F(x) - F(y)| < M\,|x - y| < M\delta = \epsilon.$$

Suppose that f is continuous at c. Also, note that, if $x \in [a, b]$ and $x < c$, then

$$\begin{aligned} \left| \frac{F(x) - F(c)}{x - c} - f(c) \right| &= \left| \frac{1}{x - c} \left(\int_a^x f(t)dt - \int_a^c f(t)dt \right) - f(c) \right| \\ &= \left| \frac{1}{x - c} \int_c^x f(t)dt - \frac{1}{x - c} \int_c^x f(c)dt \right| \\ &= \left| \frac{1}{x - c} \int_c^x [f(t) - f(c)]dt \right| \leq \frac{1}{|x - c|} \int_c^x |f(t) - f(c)|\, dt. \end{aligned}$$

Also, if $x \in [a, b]$ and $x > c$, then

$$\begin{aligned} \left| \frac{F(x) - F(c)}{x - c} - f(c) \right| &= \left| \frac{1}{x - c} \left(\int_a^x f(t)dt - \int_a^c f(t)dt \right) - f(c) \right| \\ &= \left| \frac{1}{x - c} \int_c^x f(t)dt - \frac{1}{x - c} \int_c^x f(c)dt \right| \\ &= \left| \frac{1}{x - c} \int_c^x [f(t) - f(c)]dt \right| \leq \frac{1}{|x - c|} \int_c^x |f(t) - f(c)|\, dt. \end{aligned}$$

Let $\epsilon \in \mathbb{R}_{>0}$. Since f is continuous at c, there is a $\delta \in \mathbb{R}_{>0}$ such that $|f(t) - f(c)| < \epsilon$ whenever $t \in N(c; \delta) \cap [a, b]$. Therefore, if $x \in N^*(c; \delta) \cap [a, b]$, then

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| \leq \frac{1}{|x - c|} \int_c^x |f(t) - f(c)|\, dt < \frac{1}{|x - c|} \int_c^x \epsilon dt = \epsilon.$$

□

X.

(a) Define $g(x) = \int_0^x \sqrt{9 + t^2}dt$ for all $x \in \mathbb{R}$. Since $\sqrt{9 + x^2}$ is continuous on $\mathbb{R}$, the first part of the fundamental theoem of calculus states that g is differentiable on $\mathbb{R}$. Now, the following limit

$$\lim_{x \to 0} \frac{g(x)}{x}$$

is in the indeterminate form 0/0. Applying, L'hopitals rule, we have

$$\lim_{x\to 0} \frac{g(x)}{x} = \lim_{x\to} g'(x) = \lim_{x\to} \sqrt{9+x^2} = 3.$$

(b) Let f be continuous on $[0,\infty)$. Suppose that $f(x) \neq 0$ for all $x > 0$ and that

$$f(x)^2 = 2\int_0^x f(t)dt,$$

for all $x \geq 0$. Then, $f(x) = x$ for all $x \geq 0$

Proof. Define $g(x) = \int_0^x f(t)dt$ for all $x \geq 0$. Since f is continuous on $[0,\infty)$, the first part of the fundamental theorem of calculus states that g is differentiable on $[0,\infty)$ and $g'(x) = f(x)$ for all $x \in [0,\infty)$.

Since f is continuous on $[0,\infty)$ and $f(x) \neq 0$ for all $x > 0$, the intermediate value theorem implies that f is always positive or always negative on $(0,\infty)$. If f were always negative, then $g(x) = \int_0^x f(t)dt$ would be negative for all $x \in (0,\infty)$, which would contradict $f(x)^2 = 2g(x)$. Therefore, f is always positive on $(0,\infty)$; hence, $g(x) > 0$ for all $x \in (0,\infty)$. Since $\sqrt{x}$ is differentiable on $(0,\infty)$, the chain rule states that $\sqrt{2g(x)}$ is differentiable on $(0,\infty)$. Furthermore, since f is positive on $(0,\infty)$, it follows that

$$\sqrt{f(x)^2} = |f(x)| = f(x),$$

for all $x \in (0,\infty)$. Therefore, $f(x) = \sqrt{2g(x)}$ is differentiable on $(0,\infty)$.

Since $f(x)^2 = 2g(x)$ and $g'(x) = f(x)$, the chain rule implies that

$$2f(x)f'(x) = 2f(x),$$

for all $x \in (0,\infty)$. Since $f(x) \neq 0$ for all $x \in (0,\infty)$, it follows that $f'(x) = 1$ for all $x \in (0,\infty)$. Since all antiderivatives of a function only differ by a constant, it follows that $f(x) = x + c$, for all $x \in (0,\infty)$. Note that $f(0) = 0$, since $f(0)^2 = 2g(0) = 0$. Since f is continuous on $[0,\infty)$, $\lim_{x\to 0} f(x) = f(0) = 0$. Therefore,

$$\lim_{x\to 0}(x+c) = c = 0.$$

Hence, $f(x) = x$ for all $x \geq 0$. □

(c) Let f be continuous on $[a,b]$. Suppose that

$$\int_a^x f(t)dt = \int_x^b f(t)dt,$$

for all $x \in [a,b]$. Then, $f(x) = 0$ for all $x \in [a,b]$.

Proof. For all $x \in [a,b]$, we have

$$\int_a^x f(t)dt = \int_x^b f(t)dt = \int_a^b f(t)dt - \int_a^x f(t)dt.$$

Define $g(x) = \int_a^x f(t)dt$, for all $x \in [a,b]$. Then,

$$2g(x) = \int_a^b f(t)dt,$$

for all $x \in [a,b]$. Therefore, g is constant on $[a,b]$. Since f is continuous on $[a,b]$, the first part of the fundamental theorem of calculus implies that

$$0 = g'(x) = f(x),$$

for all $x \in [a,b]$. □