Fundamental Theorem of Calculus

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1 Preliminaries

The fundamental theorem of calculus is really two theorems, each expressing that differentiation and integration are inverse operations. Historically, the operations of integration and differentiation were developed of solve seemingly unrelated problems. These problems may be described geometrically as finding the area under a curve and finding the slope of a curve at a point. The proof of their inverse relationship was one of the important theoretical (and practical) contributions of Newton and Leibniz in the seventeenth century.

Thus far, our definition of the Riemann integral $\int_a^b f(x)dx$ only holds when a < b. It will be convenient to extend this definition and let $\int_b^a f(x)dx = -\int_a^b f(x)dx$ and $\int_a^a f(x)dx = 0$.

2 The Fundamental Theorem of Calculus

The following theorem establishes the first part of the fundamental theorem of calculus.

Theorem 2.1. Let $f: [a,b] \to \mathbb{R}$ be Riemann integrable and define

$$F(x) = \int_{a}^{x} f(t)dt,$$

for each $x \in [a, b]$. Then, F is uniformly continuous on [a, b]. Furthermore, if f is continuous at $c \in (a, b)$, then F is differentiable at c and F'(c) = f(c).

Proof. Since f is Riemann integrable, it is bounded. So, there exists an $M \in \mathbb{R}_{>0}$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. Let $\epsilon \in \mathbb{R}_{>0}$ and $\delta = \epsilon/M$. Also, note that, if $x \leq y$,

$$|F(x) - F(y)| = \left| \int_{a}^{x} f(t)dt - \int_{a}^{y} f(t)dt \right|$$
$$= \left| -\int_{x}^{y} f(t)dt \right|$$
$$\leq \int_{x}^{y} |f(t)|dt \leq M |x - y|$$

Also, if x > y,

$$|F(x) - F(y)| = \left| \int_{a}^{x} f(t)dt - \int_{a}^{y} f(t)dt \right|$$
$$= \left| \int_{y}^{x} f(t)dt \right|$$
$$\leq \int_{y}^{x} |f(t)dt| \leq M |x - y|$$

Hence, for all $x, y \in [a, b]$, if $|x - y| < \delta$ the

$$|F(x) - F(y)| < M |x - y| < M\delta = \epsilon.$$

Suppose that f is continuous at c. Also, note that, if x < c, then

$$\begin{split} \left| \frac{F(x) - F(c)}{x - c} - f(c) \right| &= \left| \frac{1}{x - c} \left(\int_a^x f(t) dt - \int_a^c f(t) dt \right) - f(c) \right| \\ &= \left| \frac{1}{x - c} \int_c^x f(t) dt - \frac{1}{x - c} \int_c^x f(c) dt \right| \\ &= \left| \frac{1}{x - c} \int_c^x [f(t) - f(c)] dt \right| \le \frac{1}{|x - c|} \int_c^x |f(t) - f(c)| dt. \end{split}$$

Also, if x > c,

$$\begin{split} \left| \frac{F(x) - F(c)}{x - c} - f(c) \right| &= \left| \frac{1}{x - c} \left(\int_a^x f(t) dt - \int_a^c f(t) dt \right) - f(c) \right| \\ &= \left| \frac{1}{x - c} \int_c^x f(t) dt - \frac{1}{x - c} \int_c^x f(c) dt \right| \\ &= \left| \frac{1}{x - c} \int_c^x [f(t) - f(c)] dt \right| \leq \frac{1}{|x - c|} \int_c^x |f(t) - f(c)| dt. \end{split}$$

Let $\epsilon \in \mathbb{R}_{>0}$. Since f is continuous at c, there is a $\delta \in \mathbb{R}_{>0}$ such that $|f(t) - f(c)| < \epsilon$ whenever $t \in N(c; \delta) \cap [a, b]$. Therefore, if $x \in N^*(c; \delta) \cap [a, b]$, then

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| \le \frac{1}{|x - c|} \int_c^x |f(t) - f(c)| dt < \frac{1}{|x - c|} \int_c^x \epsilon dt = \epsilon.$$

The following theorem establishes the second part of the fundamental theorem of calculus.

Theorem 2.2. Let $f: [a,b] \to \mathbb{R}$ be Riemann integrable and let $F: [a,b] \to \mathbb{R}$ be continuous on [a,b] and differentiable on (a,b). If F'(x) = f(x) for all $x \in (a,b)$, then

$$\int_{a}^{b} f(x)dx = F(b) - F(a).$$

Proof. Let P be any partition of [a, b]. For each $i \in \{1, 2, ..., n\}$, the mean value theorem states that there is a $t_i \in (x_{i-1}, x_i)$ such that

$$F'(t_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}},$$

that is, $f(t_i)\Delta x_i = F(x_i) - F(x_{i-1})$. Since $\sum_{i=1}^n f(t_i)\Delta x_i = \sum_{i=1}^n [F(x_i) - F(x_{i-1})] = F(b) - F(a)$,

$$L(f, P) < F(b) - F(a) < U(f, P).$$

Since the above bound holds for all partitions P of [a, b],

$$L(f) \le F(b) - F(a) \le U(f).$$

Therefore, since f is Riemann integrable.

$$\int_{a}^{b} f(x)dx = L(f) = U(f) = F(b) - F(a).$$