Topology, Sequences, and Limits Homework

Real Analysis

Answer Key

1. **Topology of a set with two accumulation points**

   Let $S = \{1 - 2^{-n} : n \in \mathbb{N}\} \cup \{2 + 3^{-n} : n \in \mathbb{N}\}$.

   (a) Construct a sequence in S converging to 1 and another converging to 2. Use the accumulation-point criterion to show that $1, 2 \in S'$. Do either of these points belong to S?

   (b) Prove that S has no other accumulation points. For a point c different from the two proposed limits, first choose a neighborhood avoiding all sufficiently late terms of both sequences from part (a), then reduce its radius to exclude the finitely many remaining values other than possibly c. Determine $\text{int}(S)$, $\text{bd}(S)$, and $\text{cl}(S)$, and identify the isolated points.

   (c) Apply Heine–Borel to $\text{cl}(S)$. Define $s_{2n} = 1 - 2^{-n}$ and $s_{2n-1} = 2 + 3^{-n}$. Show that (s_n) has no subsequence converging to a point of S, although it is bounded. Explain how this distinguishes Bolzano–Weierstrass from the sequential characterization of compactness.

   Solution. The sequences $1 - 2^{-n}$ and $2 + 3^{-n}$ tend to 1 and 2, respectively, with neither limit in S. They avoid their limits, so both are accumulation points. For any other c, choose $r > 0$ smaller than a quarter of its distance from each of $1, 2$. Both tails eventually miss $N(c; r)$, and a smaller radius excludes the remaining finite set except possibly its center. Thus $S' = \{1, 2\}$, all points of S are isolated, the interior is empty, and boundary and closure both equal $S \cup \{1, 2\}$. The closure is closed and bounded, hence compact. Any convergent subsequence of (s_n) has an infinite parity subsubsequence, forcing its limit to be 1 or 2. Neither belongs to S, so S fails sequential compactness. Bolzano–Weierstrass supplies a limit in $\mathbb{R}$, not necessarily in S.

2. **Finite unions and finite subcovers**

   (a) Let $K_1, K_2 \subseteq \mathbb{R}$ be compact. Starting with an arbitrary open cover of $K_1 \cup K_2$, prove directly that their union is compact. Explain why the two finite subcollections together still consist of members of the original cover.

   (b) Let $K_n = \{1, 2, \ldots, n\}$. Each K_n is compact. Determine $\bigcup_{n=1}^{\infty} K_n$, and show that the open cover $\{(j - 1/3, j + 1/3) : j \in \mathbb{N}\}$ has no finite subcover of this union. Explain why part (a) does not extend to arbitrary infinite unions, and check the same conclusion using Heine–Borel.

   (c) Let $H \subseteq \mathbb{R}$ be nonempty and compact, and let $h : H \to \mathbb{R}$. Suppose each $x \in H$ has $r_x, B_x > 0$ such that $|h(y)| \leq B_x$ for all $y \in H \cap N(x; r_x)$. Use a finite subcover to prove that h is bounded on H. Identify the single global bound obtained from the selected local bounds.

Solution. A cover of $K_1 \cup K_2$ covers each compact set; choose a finite subcover for each and combine them. Their union remains a finite subcollection of the original cover. In (b), the union is $\mathbb{N}$. Each given interval contains exactly its center from $\mathbb{N}$, so any finite subcollection covers only finitely many natural numbers. The union is unbounded and hence not compact by Heine–Borel; infinitely many finite subcollections need not combine to a finite collection. In (c), the neighborhoods cover H. Choose finitely many with centers $x_1, \ldots, x_m$ and set $B = \max_i B_{x_i}$. Each point of H lies in one selected neighborhood and its output absolute value is at most B.

3. **A rational recurrence and an error bound**

Let $a_1 = 0$ and $a_{n+1} = (2a_n + 3)/(a_n + 4)$.

(a) Prove by induction that $0 \le a_n < 1$ for every n, so each denominator is nonzero. Derive a formula for $a_{n+1} - a_n$ and use it to prove that the sequence is increasing.

(b) Apply the monotone convergence theorem before taking limits in the recurrence. Find all roots of the resulting limit equation and use the bounds to identify the actual limit. Explain why the limit equation alone would not establish convergence.

(c) Put $e_n = 1 - a_n$. Show that $e_{n+1} = e_n/(a_n + 4) \le e_n/4$, and deduce $0 < e_n \le 4^{1-n}$. Use this estimate to give a second proof of convergence directly from the definition, choosing N so $4^{1-N} < \varepsilon$. What additional information does the estimate provide beyond the monotone convergence theorem?

Solution. If $0 \le a_n < 1$, the next numerator is positive, and $2a_n + 3 < a_n + 4$, giving $0 < a_{n+1} < 1$. The difference is $(1 - a_n)(a_n + 3)/(a_n + 4) > 0$. Monotone convergence therefore gives $L \in [0, 1]$; the recurrence yields $L^2 + 2L - 3 = (L - 1)(L + 3) = 0$, hence $L = 1$. For (c), subtract the recurrence from 1 to obtain the stated error identity. Induction from $e_1 = 1$ yields $e_n \le 4^{1-n}$. For $n \ge N$ the error is at most $4^{1-N} < \varepsilon$. The estimate quantifies the accuracy after finitely many iterations, whereas monotone convergence alone asserts existence of a limit.

4. **Cauchy sequences from interval bisection**

Set $\ell_1 = 1$, $u_1 = 2$, and $m_n = (\ell_n + u_n)/2$. If $m_n^2 < 3$, set $(\ell_{n+1}, u_{n+1}) = (m_n, u_n)$; otherwise set $(\ell_{n+1}, u_{n+1}) = (\ell_n, m_n)$.

(a) Compute m_1, m_2, m_3. Prove by induction that all endpoints and midpoints are rational, that $[\ell_{n+1}, u_{n+1}] \subseteq [\ell_n, u_n]$ contains $\sqrt{3}$, and that $u_n - \ell_n = 2^{1-n}$. You may use the irrationality of $\sqrt{3}$.

(b) For $j, k \ge N$, explain why both m_j and m_k lie in $[\ell_N, u_N]$, and deduce $|m_j - m_k| \le 2^{1-N}$. Use the Cauchy definition to prove (m_n) is Cauchy, then state the result that guarantees a real limit.

(c) Use the interval widths to show $|m_n - \sqrt{3}| \le 2^{1-n}$ and identify the limit. Explain why the same rational sequence does not converge in $\mathbb{Q}$. Identify precisely where completeness and uniqueness of limits are used.

Solution. The first midpoints are $3/2, 7/4, 13/8$. All operations preserve rationality; comparisons with 3 select the half still containing $\sqrt{3}$, and every replacement halves the width, giving 2^{1-n}. Every midpoint from index N onward lies in the nested interval $[\ell_N, u_N]$. Hence each pairwise tail distance is at most its width. Choose N with $2^{1-N} < \varepsilon$ to prove the Cauchy condition and apply completeness of $\mathbb{R}$. Because both m_n and $\sqrt{3}$ lie in that interval, their distance is at most its width and the real limit is $\sqrt{3}$. Any rational limit would also be a real limit; uniqueness would force it to equal the irrational $\sqrt{3}$. Thus $\mathbb{Q}$ is not complete.

5. **Selection and subsequential limits**

   (a) Prove that if $\lim_{k\to\infty} s_{2k} = L$ and $\lim_{k\to\infty} s_{2k-1} = L$, then $\lim_{n\to\infty} s_n = L$. Choose one tail index that controls both parity subsequences and explain why it controls every sufficiently late term of the original sequence.

   (b) Define $s_{3k} = 1 + 1/(k+1)$, $s_{3k-1} = -1 + 1/(k+1)$, and $s_{3k-2} = 2 + 1/(k+1)$ for $k \in \mathbb{N}$. Find all subsequential limits and prove that there are no others. Is (s_n) convergent? Explain why its behavior does not contradict Bolzano–Weierstrass.

   (c) Let (x_n) be bounded and suppose every convergent subsequence has limit L. Prove that the full sequence converges to L. If convergence fails, use its negation to select a subsequence satisfying $|x_{n_k} - L| \geq \varepsilon_0$ for one fixed $\varepsilon_0 > 0$; then apply Bolzano–Weierstrass to that subsequence and obtain a contradiction.

Solution. For (a), choose K so both parity errors are less than ε for all $k \geq K$. Every $n \geq 2K$ has one of these forms with $k \geq K$, so its error is less than ε. In (b), the three residue subsequences tend to $1, -1, 2$. Any convergent subsequence has infinitely many indices in at least one residue class, and its subsubsequence in that class has one of these limits; hence there are no others. Different limits imply divergence, which is compatible with boundedness and Bolzano–Weierstrass. In (c), failure of convergence gives $\varepsilon_0 > 0$ and arbitrarily late violating indices. Choose them recursively with $n_{k+1} > n_k$. The resulting subsequence is bounded, so has a convergent subsequence. That is also a subsequence of (x_n) and thus has limit L, contradicting its persistent distance at least ε_0.

6. **Compactness guarantees a closest point**

Let $K \subseteq \mathbb{R}$ be nonempty and compact, fix $p \in \mathbb{R}$, and set $d = \inf\{|x - p| : x \in K\}$.

(a) Explain why d exists and choose $x_n \in K$ with $d \leq |x_n - p| < d + 1/n$. Determine $\lim_{n\to\infty} |x_n - p|$. Use Heine–Borel, Bolzano–Weierstrass, and closedness to obtain a subsequence converging to some $x \in K$, explaining each result's role.

(b) Use $\big||x_{n_k} - p| - |x - p|\big| \leq |x_{n_k} - x|$ and uniqueness of limits to prove $|x - p| = d$. If $p \notin K$, prove $d > 0$ and find a neighborhood of p disjoint from K. Compare this conclusion with the open-complement characterization of closedness.

(c) For $K = (0, 2)$ and $p = 3$, find d and decide whether a closest point exists; identify the failed compactness hypothesis. For $K = \{-3, 1\}$ and $p = -1$, determine whether a closest point is unique. State exactly what the general argument guarantees.

Solution. The distance set is nonempty and bounded below, so has an infimum and admits the stated approximating sequence. Squeezing gives distance limit d. Compactness gives closedness and boundedness by Heine–Borel; boundedness gives a convergent subsequence by Bolzano–Weierstrass, and closedness retains its limit in K. The reverse triangle inequality gives distance limit $|x - p|$ along that subsequence, so uniqueness forces $|x - p| = d$. If $p \notin K$, then $x \neq p$, so $d > 0$ and $N(p; d)$ misses K, as an exterior neighborhood should for a closed set. For $(0, 2)$ and $p = 3$, $d = 1$ is not attained because the missing endpoint is 2; closedness fails. Both -3 and 1 are distance 2 from $p = -1$. The argument guarantees existence of a closest point, but not uniqueness.

7. **Function limits: failure, estimates, and restricted domains**

(a) Let $f : S \to \mathbb{R}$, $c \in S'$, and $L \in \mathbb{R}$. Prove the reverse implication of the sequential criterion. Suppose every sequence (x_n) in $S \setminus \{c\}$ converging to c has $\lim_{n\to\infty} f(x_n) = L$. Negate the function-limit definition if L is not the limit, choose violating inputs with $0 < |x_n - c| < 1/n$, and use a fixed output separation to obtain a contradiction.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ equal 1 on $\mathbb{Q}$ and 0 on $\mathbb{R} \setminus \mathbb{Q}$. Use two explicit sequences to show that g has no limit at 0. Nevertheless, prove directly that $\lim_{x\to 0} xg(x) = 0$. Explain why the product limit rule cannot be applied to the displayed factors, and how an estimate resolves the problem.

(c) Let $S = (-\infty, 0) \cup [1, \infty)$ and define $f(x) = 1 + x^2$ for $x < 0$ and $f(x) = 10 - x$ for $x \geq 1$. Prove directly that $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 1} f(x) = 9$, both relative to S. Identify the side from which domain points approach each point, and explain why the unavailable one-sided limit is not required.

Solution. The negation gives $\varepsilon_0 > 0$ such that every $\delta > 0$ admits a domain input with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon_0$. Choosing such inputs for $\delta = 1/n$ gives an admissible input sequence approaching c whose images do not approach L, a contradiction. In

(b), use $1/(n+1)$ and $\sqrt{2}/(n+1)$: the images are 1 and 0, respectively. However, $|xg(x)| \leq |x|$, so $\delta = \varepsilon$ proves the product limit. The factor g lacks a limit, so the product rule's hypotheses fail. In (c), at zero choose $\delta = \min\{1, \sqrt{\varepsilon}\}$. Every admissible input is negative and has output error $x^2 < \varepsilon$. At one choose $\delta = \min\{1, \varepsilon\}$, forcing admissible inputs into $[1, \infty)$ with output error $|x - 1| < \varepsilon$. The approaches are only from the left at zero and from the right at one. Each is a limit on the actual domain; neither point is an accumulation point of the domain on the unavailable side.