

Limits of Functions

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MATH 312

1 Limits and Neighborhoods

1.1 From Sequences to Functions

For a sequence, convergence describes what happens when the index is large. For a function, a limit describes what happens when the input is close to a specified point. In both cases, every prescribed error in the output must be met by a suitable restriction on the input.

Throughout these notes, $S \subseteq \mathbb{R}$, $f: S \rightarrow \mathbb{R}$, and c is an accumulation point of S , written $c \in S'$. Thus every deleted neighborhood of c contains a point of S . The point c need not belong to S .

Definition 1.1 (Limit of a function). We write $\lim_{x \rightarrow c} f(x) = L$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$x \in S \text{ and } 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

The number ϵ specifies the allowed output error. After it is given, we choose one δ that works for *every* point of the domain satisfying $0 < |x - c| < \delta$. The choice may depend on ϵ , f , and c , but not on the individual point x .

The restriction $0 < |x - c|$ excludes $x = c$. Consequently, the value $f(c)$ has no effect on the limit, even when it is defined. The condition $x \in S$ is equally important: only inputs in the domain are considered.

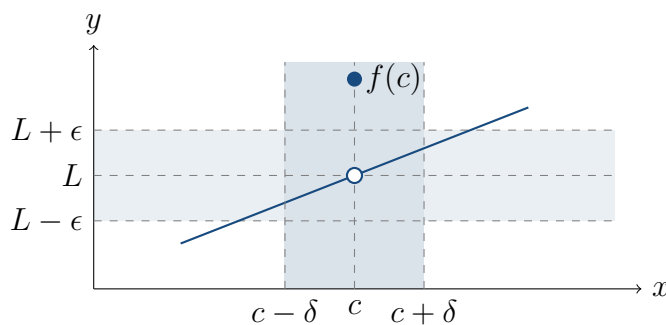


Figure 1. The limit concerns nearby inputs distinct from c . A different value at c does not change the limit.

1.2 The Neighborhood Description

Write $N(c; \delta) = (c - \delta, c + \delta)$ and $N^*(c; \delta) = N(c; \delta) \setminus \{c\}$.

Theorem 1.2 (Neighborhood criterion). *The equality $\lim_{x \rightarrow c} f(x) = L$ holds if and only if, for every neighborhood V of L , there exists $\delta > 0$ such that*

$$f(N^*(c; \delta) \cap S) \subseteq V.$$

Proof. Every neighborhood V of L contains $(L - \epsilon, L + \epsilon)$ for some $\epsilon > 0$, so the limit definition gives the required δ . Conversely, take $V = (L - \epsilon, L + \epsilon)$. $\square$

Proposition 1.3 (Uniqueness). *A function has at most one limiting value at an accumulation point of its domain.*

Proof. Suppose $L_1 \neq L_2$ are limits at c . Set $\epsilon = |L_1 - L_2|/3$. Choose $\delta_1, \delta_2 > 0$ for these limits and this error. Since $c \in S'$, there exists $x \in S$ with $0 < |x - c| < \min\{\delta_1, \delta_2\}$. Then

$$|L_1 - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < 2\epsilon = \frac{2}{3}|L_1 - L_2|,$$

which is impossible. $\square$

If c were isolated in S , a small deleted neighborhood would contain no domain points: the defining implication would hold for every L . The assumption $c \in S'$ prevents this.

2 Proving Limits Directly

2.1 Estimating the Error

To prove a proposed limit, bound $|f(x) - L|$ in terms of $|x - c|$. Choose δ to make the bound less than ϵ , then verify it for every admissible input.

Example 2.1 (A linear function). For $f(x) = 2x + 1$, prove $\lim_{x \rightarrow 3} f(x) = 7$. Given $\epsilon > 0$, choose $\delta = \epsilon/2$. Whenever $0 < |x - 3| < \delta$,

$$|f(x) - 7| = |2x - 6| = 2|x - 3| < 2\delta = \epsilon.$$

More generally, $\lim_{x \rightarrow c} x = c$ follows by choosing $\delta = \epsilon$, and a constant function $f(x) = k$ has limit k because its error is zero.

Example 2.2 (A quadratic function). For $f(x) = x^2 + 2x + 6$, prove $\lim_{x \rightarrow 3} f(x) = 21$. Factoring gives

$$|f(x) - 21| = |x + 5||x - 3|.$$

The factor $|x + 5|$ depends on x , so first restrict $|x - 3| < 1$. The triangle inequality then gives

$$|x + 5| = |(x - 3) + 8| \leq |x - 3| + 8 < 9.$$

Given $\epsilon > 0$, choose $\delta = \min\{1, \epsilon/9\}$. If $0 < |x - 3| < \delta$, both restrictions hold, and

$$|f(x) - 21| < 9|x - 3| < 9\delta \leq \epsilon.$$

Example 2.3 (A reciprocal). Prove $\lim_{x \rightarrow 2} 1/x = 1/2$, with domain $\mathbb{R} \setminus \{0\}$. We have

$$\left| \frac{1}{x} - \frac{1}{2} \right| = \frac{|x - 2|}{2|x|}.$$

If $|x - 2| < 1$, then $x > 1$, so the error is less than $|x - 2|/2$. Given $\epsilon > 0$, set $\delta = \min\{1, 2\epsilon\}$. For $0 < |x - 2| < \delta$, the error is less than $\delta/2 \leq \epsilon$. Keeping the denominator away from zero makes the estimate possible.

Example 2.4 (A removable gap). Define

$$f(x) = \begin{cases} (x^2 - 1)/(x - 1), & x \neq 1, \\ 5, & x = 1. \end{cases}$$

For $x \neq 1$, cancellation gives $f(x) = x + 1$. Thus, given $\epsilon > 0$, the choice $\delta = \epsilon$ yields $|f(x) - 2| = |x - 1| < \epsilon$ whenever $0 < |x - 1| < \delta$. Hence $\lim_{x \rightarrow 1} f(x) = 2$, even though $f(1) = 5$.

Changing $f(1)$, or leaving it undefined, does not change this limit. Functions that agree on a deleted neighborhood of c within a common domain have the same limit behavior at c .

3 The Sequential Criterion

3.1 Testing Every Approach

The following theorem connects the two notions of limit: controlling *every* sequence of inputs approaching c is equivalent to controlling all nearby domain points.

Theorem 3.1 (Sequential criterion). Let $f: S \rightarrow \mathbb{R}$, $c \in S'$, and $L \in \mathbb{R}$. Then $\lim_{x \rightarrow c} f(x) = L$ if and only if, for every sequence (s_n) with $s_n \in S \setminus \{c\}$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} s_n = c$, we have

$$\lim_{n \rightarrow \infty} f(s_n) = L.$$

Proof. Suppose $\lim_{x \rightarrow c} f(x) = L$, and let (s_n) satisfy the stated conditions. Given $\epsilon > 0$, choose $\delta > 0$ from the function-limit definition. Since $\lim_{n \rightarrow \infty} s_n = c$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - c| < \delta$. Also $s_n \neq c$, so $|f(s_n) - L| < \epsilon$ for every $n \geq N$. This proves the sequence limit.

For the converse, suppose the function does not have limit L at c . Negating the definition gives a fixed $\epsilon_0 > 0$ such that, for every $\delta > 0$, there exists $x \in S$ with

$$0 < |x - c| < \delta \quad \text{and} \quad |f(x) - L| \geq \epsilon_0.$$

For each $n \in \mathbb{N}$, choose such a point s_n with $\delta = 1/n$. Then $0 < |s_n - c| < 1/n$, so $\lim_{n \rightarrow \infty} s_n = c$, but $|f(s_n) - L| \geq \epsilon_0$ for every n . Thus $(f(s_n))$ does not converge to L , contradicting the stated sequence condition. $\square$

3.2 Showing That a Limit Does Not Exist

One successful sequence does not prove a function limit. To disprove a proposed value L , one admissible sequence whose images do not converge to L suffices. To rule out *every* real

limit, it is often convenient to find two sequences approaching the same point with different image limits.

Example 3.1 (A jump). Let $f(x) = |x|/x$ for $x \neq 0$. Set $s_n = 1/n$ and $t_n = -1/n$. Both sequences converge to 0, but $f(s_n) = 1$ and $f(t_n) = -1$ for all n . A function limit would have to equal both 1 and -1 . Therefore f has no limit at 0.

Example 3.2 (Persistent oscillation). Let $f(x) = \sin(1/x)$ for $x \neq 0$. Choose

$$s_n = \frac{1}{\pi/2 + 2\pi n}, \quad t_n = \frac{1}{3\pi/2 + 2\pi n}.$$

Both input sequences converge to 0, whereas $f(s_n) = 1$ and $f(t_n) = -1$. Thus no limit exists at 0, even when inputs are restricted to be positive. Boundedness of the function does not force convergence.

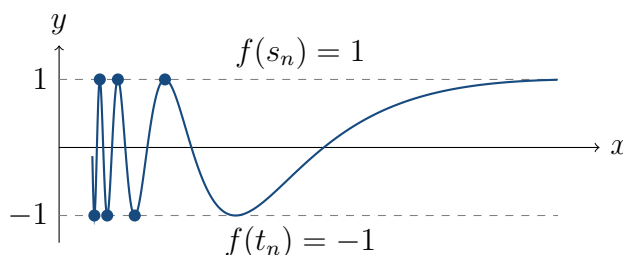


Figure 2. Oscillation retains its full size near 0. Two sequences detect incompatible limiting values.

4 Limit Theorems and Applications

4.1 Algebraic Rules

The sequential criterion allows us to transfer the limit theorems for sequences to functions. The domain and accumulation-point assumptions remain in force.

Theorem 4.1 (Algebraic limit rules). Suppose $f, g: S \rightarrow \mathbb{R}$, $c \in S'$, $\lim_{x \rightarrow c} f(x) = L$, and $\lim_{x \rightarrow c} g(x) = K$. For every $a \in \mathbb{R}$,

- (a) $\lim_{x \rightarrow c} (f(x) + g(x)) = L + K$;
- (b) $\lim_{x \rightarrow c} (af(x)) = aL$;
- (c) $\lim_{x \rightarrow c} (f(x)g(x)) = LK$;
- (d) $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$.

In part (d), the quotient is defined on $T = \{x \in S : g(x) \neq 0\}$, which contains all points of S sufficiently close to c and distinct from c .

Proof. For any sequence (s_n) in $S \setminus \{c\}$ converging to c , the sequential criterion gives $\lim_{n \rightarrow \infty} f(s_n) = L$ and $\lim_{n \rightarrow \infty} g(s_n) = K$. The corresponding sequence rules and Theorem 3.1 prove (a)–(c).

If $K \neq 0$, choose $\delta_0 > 0$ so that $0 < |x - c| < \delta_0$ and $x \in S$ imply $|g(x) - K| < |K|/2$. Then

$$|g(x)| \geq |K| - |g(x) - K| > |K|/2 > 0.$$

Hence $c \in T'$. Apply the sequence quotient rule to every sequence in $T \setminus \{c\}$ converging to c , and use the sequential criterion on T . $\square$

Example 4.1 (Polynomials and rational functions). Since $\lim_{x \rightarrow c} x = c$, repeated products and sums give $\lim_{x \rightarrow c} p(x) = p(c)$ for every polynomial p . Thus, if p, q are polynomials and $q(c) \neq 0$,

$$\lim_{x \rightarrow c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.$$

For example, $\lim_{x \rightarrow 2} (x^2 + 3)/(x + 1) = 7/3$. If the denominator has limit zero, the quotient rule makes no assertion. In Example 2.4, cancellation was needed before evaluating the limit.

4.2 Order and Squeezing

Proposition 4.2 (Order preservation). Suppose $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = K$. If there exists $r > 0$ such that $f(x) \leq g(x)$ for all $x \in S$ with $0 < |x - c| < r$, then $L \leq K$.

Proof. Choose $s_n \in S$ with $0 < |s_n - c| < \min\{r, 1/n\}$. The sequential criterion and order preservation for sequences give $L \leq K$. $\square$

Strict inequalities may become equality in the limit: for $x \neq 0$, $0 < x^2$, but both functions have limit 0 at 0.

Theorem 4.3 (Squeeze theorem). Let $f, g, h: S \rightarrow \mathbb{R}$, $c \in S'$, and suppose there exists $r > 0$ such that

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \in S \text{ with } 0 < |x - c| < r.$$

If $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$, then $\lim_{x \rightarrow c} g(x) = L$.

Proof. For every admissible sequence (s_n) converging to c , the inequalities hold for all $n \geq N$ for some $N \in \mathbb{N}$. Apply the squeeze theorem for sequences and then the sequential criterion. $\square$

Example 4.2 (Oscillation with shrinking size). For $x \neq 0$, $-|x| \leq x \sin(1/x) \leq |x|$. The inequality $||x| - |c|| \leq |x - c|$ proves $\lim_{x \rightarrow c} |x| = |c|$ directly. Thus the squeeze theorem gives

$$\lim_{x \rightarrow 0} x \sin(1/x) = 0.$$

Equivalently, $|x \sin(1/x)| \leq |x|$ gives the direct choice $\delta = \epsilon$. The product rule cannot be used with the two displayed factors, because $\sin(1/x)$ has no limit at 0.

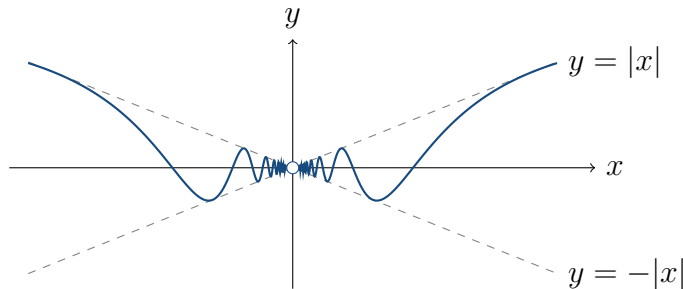


Figure 3. The shrinking envelopes control the output error despite repeated oscillation.

4.3 One-Sided Limits

A one-sided limit is a limit on a restricted domain. Write $S_+ = S \cap (c, \infty)$ and $S_- = S \cap (-\infty, c)$. If $c \in S'_+$, the notation $\lim_{x \rightarrow c^+} f(x) = L$ means that, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$x \in S \text{ and } 0 < x - c < \delta \implies |f(x) - L| < \epsilon.$$

For the left-hand limit $\lim_{x \rightarrow c^-} f(x) = L$, require $c \in S'_-$ and replace the input restriction by $0 < c - x < \delta$.

Proposition 4.4 (Combining the two sides). *If c is an accumulation point of both S_+ and S_- , then $\lim_{x \rightarrow c} f(x) = L$ if and only if*

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L.$$

Proof. A limit on S controls the inputs on either side. Conversely, given $\epsilon > 0$, take the minimum of the two one-sided choices of δ . Every admissible input distinct from c lies on one of the two sides, so its output error is less than ϵ . $\square$

Example 4.3 (A jump and an endpoint). For $f(x) = |x|/x$, the left-hand limit at 0 is -1 and the right-hand limit is 1 . Their disagreement rules out a limit on $\mathbb{R} \setminus \{0\}$.

For $g(x) = \sqrt{x}$ on $S = [0, \infty)$, only the right-hand approach is available at 0. Given $\epsilon > 0$, choose $\delta = \epsilon^2$. If $x \in S$ and $0 < |x| < \delta$, then $|\sqrt{x} - 0| < \epsilon$. Thus $\lim_{x \rightarrow 0} g(x) = 0$ relative to its domain, also written $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$. A left-hand limit is not required at this endpoint.

4.4 Local Bounds and the Next Step

Proposition 4.5 (Local boundedness and sign). *If $\lim_{x \rightarrow c} f(x) = L \in \mathbb{R}$, then there exist $r > 0$ and $B > 0$ such that $|f(x)| < B$ for all $x \in S$ with $0 < |x - c| < r$. If $L \neq 0$, then $f(x)$ has the same sign as L on some deleted neighborhood of c within S .*

Proof. Use $\epsilon = 1$ to obtain $|f(x)| \leq |f(x) - L| + |L| < 1 + |L|$. If $L \neq 0$, use $\epsilon = |L|/2$. The interval $(L - |L|/2, L + |L|/2)$ lies entirely on the same side of zero as L . $\square$

For example, $1/x^2$ has no finite limit at 0: along $s_n = 1/n$, its values are n^2 . Unlike $\sin(1/x)$, it is not bounded on any deleted neighborhood of 0.

For $c \in S \cap S'$, *continuity at c* means $\lim_{x \rightarrow c} f(x) = f(c)$. In Example 2.4, assigning $f(1) = 2$ makes the function continuous at 1.