

Monotone and Cauchy Sequences

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MATH 312

1 Monotone Sequences

1.1 Direction and Bounds

Definition 1.1. A sequence (s_n) is *increasing* (or *nondecreasing*) if $s_n \leq s_{n+1}$ for every $n \in \mathbb{N}$, and *decreasing* (or *nonincreasing*) if $s_n \geq s_{n+1}$ for every $n \in \mathbb{N}$. It is *monotone* if it is increasing or decreasing. Strictly increasing and strictly decreasing sequences satisfy the corresponding strict inequalities.

Constant sequences are both increasing and decreasing. For an increasing sequence, repeated comparison gives $s_m \leq s_n$ whenever $m \leq n$.

Example 1.2 (Different behaviors). The familiar sequence $s_n = n/(n+1)$ is strictly increasing because

$$s_{n+1} - s_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0.$$

It is bounded: $0 < s_n < 1$. The sequence $(1/n)$ is decreasing and bounded, whereas (n) is increasing and unbounded.

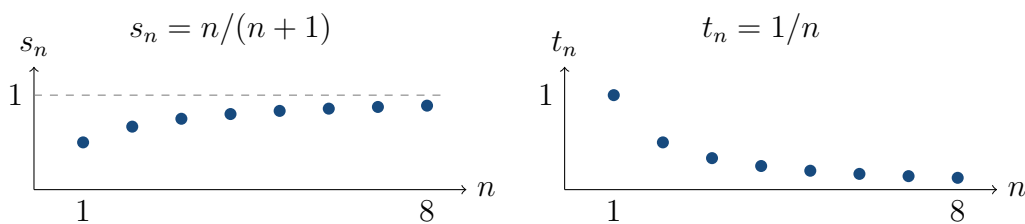


Figure 1. Bounded monotone sequences approach their limits from one side.

Example 1.3 (A decreasing sequence with negative terms). For $u_n = -n/(n+1)$, we have $u_{n+1} - u_n = -1/((n+1)(n+2)) < 0$ and $-1 < u_n < 0$. Decreasing describes order, not sign.

1.2 The Monotone Convergence Theorem

An increasing sequence is bounded below by its first term. Thus only an upper bound is needed to make it bounded. For a decreasing sequence, only a lower bound is needed.

Theorem 1.4 (Monotone convergence theorem). *A monotone real sequence converges if and only if it is bounded. More precisely:*

- (a) *If (s_n) is increasing and bounded above, then $\lim_{n \rightarrow \infty} s_n = \sup\{s_n : n \in \mathbb{N}\}$.*
- (b) *If (s_n) is decreasing and bounded below, then $\lim_{n \rightarrow \infty} s_n = \inf\{s_n : n \in \mathbb{N}\}$.*

Proof. (a) The set $S = \{s_n : n \in \mathbb{N}\}$ is nonempty and bounded above, so completeness gives $L = \sup S \in \mathbb{R}$. Let $\epsilon > 0$. Since $L - \epsilon < L$, the number $L - \epsilon$ is not an upper bound for S . Therefore some $N \in \mathbb{N}$ satisfies $s_N > L - \epsilon$. Monotonicity and the upper bound L give

$$L - \epsilon < s_N \leq s_n \leq L < L + \epsilon \quad (n \geq N).$$

Hence $|s_n - L| < \epsilon$ for every $n \geq N$.

(b) Set $L = \inf S$. Given $\epsilon > 0$, the number $L + \epsilon$ is not a lower bound, so some N satisfies $s_N < L + \epsilon$. For $n \geq N$,

$$L - \epsilon < L \leq s_n \leq s_N < L + \epsilon.$$

Thus $\lim_{n \rightarrow \infty} s_n = L$. Conversely, every convergent sequence is bounded, as proved in the sequence notes. $\square$

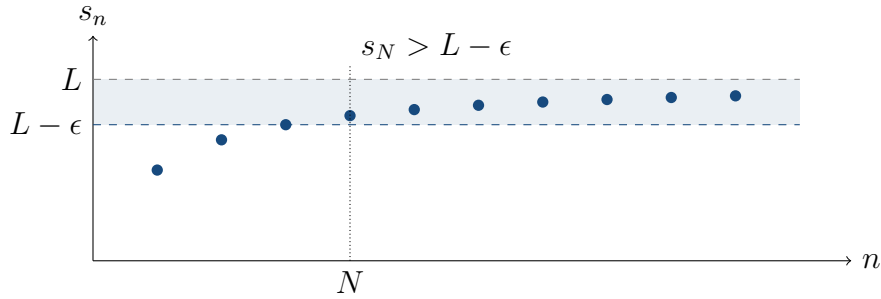


Figure 2. The supremum supplies one term near L ; monotonicity keeps all later terms near L . Here $s_n = n/(n + 1)$, $\epsilon = 1/4$, and $N = 4$.

Proposition 1.5 (Unbounded monotone sequences). *An increasing sequence unbounded above satisfies $\lim_{n \rightarrow \infty} s_n = +\infty$. A decreasing sequence unbounded below satisfies $\lim_{n \rightarrow \infty} s_n = -\infty$.*

Proof. For the increasing case, given $A \in \mathbb{R}$, unboundedness supplies N with $s_N > A$. Then $s_n \geq s_N > A$ for every $n \geq N$. The decreasing case follows by reversing the inequalities. $\square$

These are statements of divergence to infinity, not convergence to a real number.

1.3 Recursively Defined Sequences

A recurrence can suggest an equation for a limit, but that equation does not prove convergence. First establish bounds and monotonicity; then use limit laws to identify the limit.

Example 1.6 (A nested radical). Let $s_1 = 1$ and $s_{n+1} = \sqrt{1 + s_n}$.

Bounds. By induction, $1 \leq s_n \leq 2$ for every n . The claim holds for s_1 . If $1 \leq s_n \leq 2$, then $1 \leq \sqrt{1 + s_n} \leq \sqrt{3} < 2$, proving the induction step and ensuring that each term is defined.

Monotonicity. We have $s_2 = \sqrt{2} > s_1$. If $s_{n+1} > s_n$, the order-preserving property of the square root gives

$$s_{n+2} = \sqrt{1 + s_{n+1}} > \sqrt{1 + s_n} = s_{n+1}.$$

Thus (s_n) is increasing and bounded, so Theorem 1.4 gives a limit L with $1 \leq L \leq 2$.

The limit. If $\lim_{n \rightarrow \infty} s_n = L$, then $\lim_{n \rightarrow \infty} s_{n+1} = L$: the same tail estimate works after shifting the index. Squaring the recurrence avoids needing a square-root limit rule:

$$s_{n+1}^2 = 1 + s_n \implies L^2 = 1 + L.$$

Since $L \geq 1$, the positive root is required: $\lim_{n \rightarrow \infty} s_n = (1 + \sqrt{5})/2$.

Example 1.7 (Approximating $\sqrt{2}$). Let $x_1 = 2$ and $x_{n+1} = \frac{1}{2}(x_n + 2/x_n)$.

We first prove $x_n \geq \sqrt{2}$ by induction. This holds for x_1 . If $x_n \geq \sqrt{2} > 0$, then the next term is defined and

$$x_{n+1} - \sqrt{2} = \frac{x_n^2 + 2 - 2\sqrt{2}x_n}{2x_n} = \frac{(x_n - \sqrt{2})^2}{2x_n} \geq 0.$$

Thus the lower bound holds for every n . Consequently,

$$x_{n+1} - x_n = \frac{2 - x_n^2}{2x_n} \leq 0,$$

so (x_n) is decreasing. The monotone convergence theorem gives a limit $L \geq \sqrt{2} > 0$. Applying the limit laws to the recurrence yields $L = \frac{1}{2}(L + 2/L)$, hence $L^2 = 2$ and $L = \sqrt{2}$.

Example 1.8 (A monotone tail). Let $a_1 = 10$ and $a_n = 1 - 1/n$ for $n \geq 2$. Its tail is increasing and bounded above by 1, so $\lim_{n \rightarrow \infty} a_n = 1$. The first term does not affect convergence, but $\sup\{a_n : n \in \mathbb{N}\} = 10$. For a monotone tail, apply the theorem to that tail rather than to the full range.

2 Cauchy Sequences

2.1 The Definition and Basic Properties

Definition 2.1. A sequence (s_n) is *Cauchy* if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon$ for all $m, n \geq N$.

One N must work for *every pair* of late indices, not just consecutive indices. Failure means that some $\epsilon_0 > 0$ satisfies: for every N , there are $m, n \geq N$ with $|s_n - s_m| \geq \epsilon_0$.

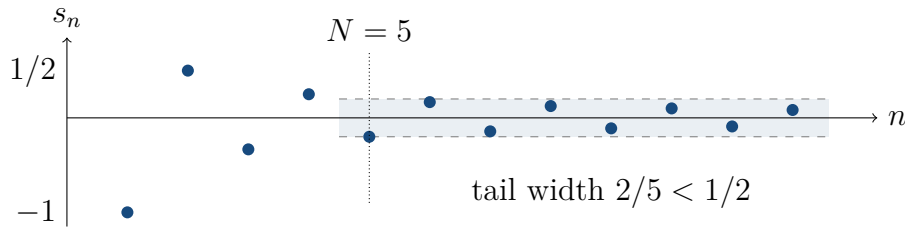


Figure 3. Cauchy sequences may oscillate. The entire tail must become narrow, not just the gaps between consecutive terms.

Example 2.2 (A direct estimate). For $s_n = (-1)^n/n$ and $m, n \geq N$, we have $|s_n - s_m| \leq 1/n + 1/m \leq 2/N$. Choosing $N > 2/\epsilon$ proves that (s_n) is Cauchy without naming its limit.

Proposition 2.3. *Every convergent sequence is Cauchy.*

Proof. If $\lim_{n \rightarrow \infty} s_n = L$, choose N so that $|s_k - L| < \epsilon/2$ for $k \geq N$. Then for $m, n \geq N$, $|s_n - s_m| \leq |s_n - L| + |s_m - L| < \epsilon$. $\square$

Proposition 2.4. *Every Cauchy sequence is bounded.*

Proof. Choose N so that $|s_n - s_m| < 1$ for $m, n \geq N$. Setting $m = N$ gives $|s_n| < 1 + |s_N|$ for $n \geq N$. The finite initial segment is bounded, so $B = \max(\{1 + |s_N|\} \cup \{|s_n| : 1 \leq n < N\})$ bounds every term. $\square$

2.2 Small Consecutive Differences Are Not Enough

Example 2.5 (The harmonic partial sums). Let $h_n = \sum_{k=1}^n 1/k$. Its consecutive differences tend to zero, since $h_{n+1} - h_n = 1/(n+1)$. However,

$$h_{2n} - h_n = \sum_{k=n+1}^{2n} \frac{1}{k} \geq n \cdot \frac{1}{2n} = \frac{1}{2}.$$

For every N , the indices $n = N$ and $m = 2N$ violate the Cauchy condition with $\epsilon_0 = 1/2$. Thus (h_n) does not converge: small steps can accumulate into a large change.

3 The Cauchy Criterion and Completeness

3.1 From Nearby Terms to a Limit

Compactness supplies an accumulation point; the Cauchy condition turns it into a limit.

Theorem 3.1 (Bolzano–Weierstrass, set form). *Every bounded infinite subset of $\mathbb{R}$ has an accumulation point in $\mathbb{R}$.*

Proof. Let S be bounded and infinite, and choose $B > 0$ with $S \subseteq [-B, B]$. Suppose S has no accumulation point in $\mathbb{R}$. For each $x \in [-B, B]$, there is $r_x > 0$ such that $(x - r_x, x + r_x)$ contains no point of S other than possibly x . These open intervals cover the compact set $[-B, B]$, so finitely many cover it. Each contains at most one point of S , forcing S to be finite, a contradiction. $\square$

Theorem 3.2 (Cauchy convergence criterion). *A sequence of real numbers converges if and only if it is Cauchy.*

Proof. The forward implication is Proposition 2.3. Conversely, let (s_n) be Cauchy and set $S = \{s_n : n \in \mathbb{N}\}$. Proposition 2.4 shows that S is bounded.

If S has one element, the sequence is constant. Otherwise, if S is finite, the smallest distance δ between distinct elements is positive. Choose N so that $|s_n - s_m| < \delta$ for $m, n \geq N$. Then $s_n = s_N$ for $n \geq N$, so the sequence converges.

If S is infinite, Theorem 3.1 gives an accumulation point L . Every neighborhood of L contains infinitely many distinct points of S : otherwise a smaller neighborhood could exclude all points other than L , contradicting accumulation.

Let $\epsilon > 0$, and choose N so that $|s_n - s_m| < \epsilon/2$ for all $m, n \geq N$. The interval $(L - \epsilon/2, L + \epsilon/2)$ contains infinitely many distinct points of S , while $s_1, \dots, s_{N-1}$ supply only finitely many values. Therefore some $m \geq N$ satisfies $|s_m - L| < \epsilon/2$. Fix this index m . For every $n \geq N$,

$$|s_n - L| \leq |s_n - s_m| + |s_m - L| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $\lim_{n \rightarrow \infty} s_n = L$. $\square$

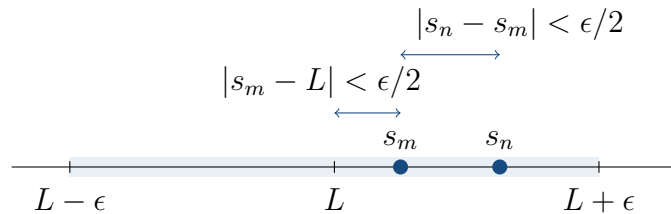


Figure 4. One term near an accumulation point anchors the Cauchy tail inside $(L - \epsilon, L + \epsilon)$. Positions are illustrative.

3.2 Why the Ambient Space Matters

The estimates defining a Cauchy sequence involve only distances between its terms. Whether the sequence converges also depends on which possible limits belong to the space.

Example 3.3 (Rational approximations to a missing limit). Let $\lfloor x \rfloor$ denote the greatest integer at most x , and define

$$q_n = \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n} \in \mathbb{Q}.$$

The first terms are 1.4, 1.41, 1.414, $\dots$. From the definition of the floor,

$$0 \leq \sqrt{2} - q_n < 10^{-n}.$$

Thus $\lim_{n \rightarrow \infty} q_n = \sqrt{2}$ as a sequence in $\mathbb{R}$. Also, for $m, n \geq N$,

$$|q_n - q_m| \leq |q_n - \sqrt{2}| + |q_m - \sqrt{2}| < 10^{-n} + 10^{-m} \leq 2 \cdot 10^{-N},$$

so (q_n) is Cauchy using the usual distance on $\mathbb{Q}$. Yet it has no limit in $\mathbb{Q}$: any such limit would also be a real limit, contradicting uniqueness and the irrationality of $\sqrt{2}$.

Definition 3.4. A metric space is *complete* if every Cauchy sequence in that space converges to a point of the space.

The Cauchy criterion says that $\mathbb{R}$ with its usual distance is complete; the preceding example shows that $\mathbb{Q}$ is not. Earlier, rational Cauchy sequences were used to construct real numbers. Here the completeness of $\mathbb{R}$ ensures that every real Cauchy sequence has a real limit. The least-upper-bound property enters the monotone theorem directly and the Cauchy theorem through compactness and Bolzano–Weierstrass.

Example 3.5 (A direct Cauchy estimate). Let $a_n = n/(n+1)$. Since $a_n = 1 - 1/(n+1)$, for $m, n \geq N$,

$$|a_n - a_m| = \left| \frac{1}{m+1} - \frac{1}{n+1} \right| \leq \frac{1}{m+1} + \frac{1}{n+1} \leq \frac{2}{N+1}.$$

Given $\epsilon > 0$, choose $N > 2/\epsilon$. Thus (a_n) is Cauchy, so the criterion gives convergence. The limit laws identify its limit as 1.

Example 3.6 (A Cauchy sequence with a candidate limit). Suppose (s_n) is Cauchy and $s_{2k} = 2 + 1/k$ for every $k \in \mathbb{N}$. Although the odd terms are unspecified, the entire sequence must converge to 2.

Given $\epsilon > 0$, choose N so that $|s_n - s_m| < \epsilon/2$ for $m, n \geq N$. Choose k with $2k \geq N$ and $1/k < \epsilon/2$. For every $n \geq N$,

$$|s_n - 2| \leq |s_n - s_{2k}| + |s_{2k} - 2| < \epsilon/2 + 1/k < \epsilon.$$

Hence $\lim_{n \rightarrow \infty} s_n = 2$. One late even term near 2 anchors all late terms, including the odd ones.

Reading. Steven R. Lay, *Analysis with an Introduction to Proof*, Section 18 (monotone sequences and Cauchy sequences), with the set form of Bolzano–Weierstrass from Section 14.