

Natural Numbers Worksheet

MATH 312: Real Analysis

Answer Key

1. Set construction and recursive arithmetic.

Recall that $0 = \emptyset$, $S(n) = n \cup \{n\}$, and

$$n + 0 = n, \quad n + S(m) = S(n + m),$$

$$n \cdot 0 = 0, \quad n \cdot S(m) = n \cdot m + n.$$

- (a) Starting with $0 = \emptyset$, write 1, 2, 3, and 4 explicitly as sets. Your final expression for 4 should also make clear why $4 = \{0, 1, 2, 3\}$.
- (b) Use only the recursive definition of addition and the successor operation to evaluate $3 + 2$. Show each recursive step.
- (c) Use the recursive definitions of multiplication and addition to evaluate $3 \cdot 2$. Show where the multiplication recursion ends and the addition recursion begins.

Solution.

(a)

$$1 = S(0) = \{\emptyset\} = \{0\},$$

$$2 = S(1) = 1 \cup \{1\} = \{0, 1\}, \quad 3 = S(2) = 2 \cup \{2\} = \{0, 1, 2\},$$

and

$$4 = S(3) = 3 \cup \{3\} = \{0, 1, 2\} \cup \{3\} = \{0, 1, 2, 3\}.$$

(b)

$$3 + 2 = 3 + S(1) = S(3 + 1) = S(3 + S(0)) = S(S(3 + 0)) = S(S(3)) = 5.$$

(c) First use the multiplication recursion:

$$3 \cdot 2 = 3 \cdot S(1) = 3 \cdot 1 + 3 = 3 \cdot S(0) + 3 = 3 \cdot 0 + 3 + 3 = 0 + 3 + 3 = 6.$$

2. A standard induction proof.

Prove that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for every $n \in \mathbb{N}$.

Your proof should clearly identify the statement $P(n)$, the basis for induction, the induction hypothesis, and the induction step. In the induction step, explain why the next odd number is $2k + 1$.

Solution. Let $P(n)$ be the statement

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

For $n = 1$, the left side is 1, and $1 = 1^2$. Thus $P(1)$ is true.

Fix $k \in \mathbb{N}$ and suppose that

$$1 + 3 + \cdots + (2k - 1) = k^2.$$

The next odd number is $2(k + 1) - 1 = 2k + 1$. Therefore,

$$\begin{aligned} 1 + 3 + \cdots + (2k - 1) + (2k + 1) &= k^2 + (2k + 1) \\ &= (k + 1)^2. \end{aligned}$$

Hence $P(k)$ implies $P(k + 1)$. By mathematical induction, $P(n)$ is true for every $n \in \mathbb{N}$.

3. Deriving distributivity from the recursive definitions.

Fix $a, b \in \mathbb{N}$. Use induction on c and the recursive definitions

$$x + S(y) = S(x + y) \quad \text{and} \quad x \cdot S(y) = x \cdot y + x$$

to prove that

$$a(b + c) = ab + ac$$

for every $c \in \mathbb{N}$.

You may use associativity of addition and the identity $0 + x = x$. Your proof should:

- (a) verify the basis $c = 1$, including the identity $a \cdot 1 = a$;
- (b) state the induction hypothesis for $c = k$;
- (c) begin the induction step with $a(b + S(k))$, and justify each equality.

Solution. Let $P(c)$ be the statement

$$a(b + c) = ab + ac.$$

First,

$$a \cdot 1 = a \cdot S(0) = a \cdot 0 + a = 0 + a = a.$$

Therefore,

$$a(b + 1) = a \cdot S(b) = ab + a = ab + a \cdot 1,$$

so $P(1)$ is true.

Now fix $k \in \mathbb{N}$ and suppose that

$$a(b + k) = ab + ak.$$

Using the recursive definitions, the induction hypothesis, and associativity of addition,

$$\begin{aligned} a(b + S(k)) &= a(S(b + k)) \\ &= a(b + k) + a \\ &= (ab + ak) + a \\ &= ab + (ak + a) \\ &= ab + aS(k). \end{aligned}$$

Thus $P(k)$ implies $P(S(k))$, which is $P(k + 1)$. By mathematical induction, $a(b + c) = ab + ac$ for every $c \in \mathbb{N}$.