# Real Analysis

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## 1 Daily Quiz

Define a sequence that is

a. monotone but not Cauchy,

b. Cauchy but not monotone.

## 2 Key Topics

Today we discuss limits of functions. For further reading, see [1, Section 3.1]. Note that [1] refers to an accumulation point as a cluster point.

Recall that $c \in \mathbb{R}$ is an accumulation point of $S \subseteq \mathbb{R}$ if

$$\forall \epsilon > 0,\ N^*(c;\epsilon) \cap S \neq \emptyset.$$

The following proposition introduces an equivalent definition of an accumulation point of $S$.

**Proposition 2.1.** *Let $S \subseteq \mathbb{R}$. Then, $c$ is an accumulation point of $S$ if and only if there exists a convergent sequence $s\colon \mathbb{N} \to \mathbb{R}$ such that* $\operatorname{rng}(s) \subseteq S \setminus \{c\}$ *and* $\lim_{n\to\infty} s_n = c$.

*Proof.* Suppose that $c$ is an accumulation point of $S$. For each $n \in \mathbb{N}$, pick $s_n \in N^*(c;1/n) \cap S$. Let $\epsilon > 0$ and define $N = 1/\epsilon$ so that

$$n > N \Rightarrow |c - s_n| < \frac{1}{n} < \epsilon.$$

Conversely, suppose that $s\colon \mathbb{N} \to \mathbb{R}$ is a sequence such that $\operatorname{rng}(s) \subseteq S \setminus \{c\}$ and $\lim_{n\to\infty} s_n = c$. Let $\epsilon > 0$. Then, there is a $N \in \mathbb{R}$ such that

$$n > N \Rightarrow |s_n - c| < \epsilon \Rightarrow s_n \in N^*(c;\epsilon) \cap S.$$

Therefore, $c$ is an accumulation point of $S$. □

### 2.1 Limits of Functions

**Definition 2.2.** Let $f\colon S \to \mathbb{R}$, $L \in \mathbb{R}$, and $c$ be an accumulation point of $S$. We say that $f$ *converges* to $L$ as $x$ approaches $c$ if

$$\forall \epsilon > 0,\ \exists \delta > 0 \ \ni\ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

If $f$ converges to $L$ as $x$ approaches $c$, we write

$$\lim_{x\to c} f(x) = L.$$

**Proposition 2.3.** *Let $f\colon S \to \mathbb{R}$ and let $c$ be an accumulation point of $S$. If $f$ converges as $x$ approaches $c$, then the limiting value is unique.*

*Proof.* Suppose that $L, L' \in \mathbb{R}$ satisfy

$$\lim_{x \to c} f(x) = L \text{ and } \lim_{x \to c} f(x) = L'.$$

Let $\epsilon > 0$. Then, there exists $\delta_1, \delta_2 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}$$

and

$$0 < |x - c| < \delta_2 \Rightarrow |f(x) - L'| < \frac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, $0 < |x - c| < \delta$ implies that

$$\begin{aligned} |L - L'| &= |L - f(x) + f(x) - L'| \\ &\leq |f(x) - L| + |f(x) - L'| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since the above inequality holds for any $\epsilon > 0$, it follows that $L = L'$. □

### 2.2 Sequential Limits

**Theorem 2.4.** *Let $f \colon S \to \mathbb{R}$, $L \in \mathbb{R}$, and $c$ be an accumulation point of $S$. Then, $\lim_{x \to c} f(x) = L$ if and only if for every sequence $s \colon \mathbb{N} \to \mathbb{R}$ such that $\operatorname{rng}(s) \subseteq S \setminus \{c\}$ and $\lim_{n \to \infty} s_n = c$ we have*

$$\lim_{n \to \infty} f(s_n) = L$$

*Proof.* Suppose that $\lim_{x \to c} f(x) = L$ and let $s \colon \mathbb{N} \to \mathbb{R}$ such that $\operatorname{rng}(s) \subseteq S \setminus \{c\}$ and $\lim_{n \to \infty} s_n = c$. Let $\epsilon > 0$. Then, there is a $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

Also, there is a $N \in \mathbb{R}$ such that

$$n > N \Rightarrow 0 < |s_n - c| < \delta.$$

Therefore, we have

$$n > N \Rightarrow 0 < |s_n - c| < \delta \Rightarrow |f(s_n) - L| < \epsilon,$$

which implies that $\lim_{n \to \infty} f(s_n) = L$.

Conversely, suppose that $\lim_{x \to c} f(x) \neq L$. Then, there exists a $\epsilon > 0$ such that for all $\delta > 0$ there exists an $x \in S$ such that $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$. In particular, for each $n \in \mathbb{N}$ there exists a $s_n \in S$ such that $0 < |s_n - c| < 1/n$ and $|f(s_n) - L| \geq \epsilon$. Therefore, $\lim_{n \to \infty} s_n = c$ and $\lim_{n \to \infty} f(s_n) \neq L$. □

Using Theorem 2.4, we can start applying everything we know about sequential limits to limits of functions, e.g., see the Limit Theorems from October 2, 2023.

## 3 Exercises

I. Prove Proposition 2.1

II. Prove Theorem 2.4.

III. Use Theorem 2.4 to show that $f(x) = \sin(1/x)$ does not converge as $x$ approaches 0.

IV. Use Theorem 2.4 to show that $\lim_{x \to 0} x \sin(1/x)$ does converge as $x$ approaches 0.

## References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.