Real Analysis

Thomas R. Cameron

October 13, 2023

1 Daily Quiz

Let $f\colon S \to \mathbb{R}$, $L \in \mathbb{R}$, and c be an accumulation point of S. State the definition of $\lim_{x\to c} f(x) = L$ and its negation $\lim_{x\to c} f(x) \neq L$.

2 Key Topics

Today we finish our discussion of sequential limits. For further reading, see [1, Section 3.1]. Note that [1] refers to an accumulation point as a cluster point.

We begin by completing the proof of the following theorem.

Theorem 2.1. *Let $f\colon S \to \mathbb{R}$, $L \in \mathbb{R}$, and c be an accumulation point of S. Then, $\lim_{x\to c} f(x) = L$ if and only if for every sequence $s\colon \mathbb{N} \to \mathbb{R}$ such that $\operatorname{rng}(s) \subseteq S \setminus \{c\}$ and $\lim_{n\to\infty} s_n = c$ we have*

$$\lim_{n\to\infty} f(s_n) = L$$

Proof. Suppose that $\lim_{x\to c} f(x) = L$ and let $s\colon \mathbb{N} \to \mathbb{R}$ such that $\operatorname{rng}(s) \subseteq S \setminus \{c\}$ and $\lim_{n\to\infty} s_n = c$. Let $\epsilon > 0$. Then, there is a $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

Also, there is a $N \in \mathbb{R}$ such that

$$n > N \Rightarrow 0 < |s_n - c| < \delta.$$

Therefore, we have

$$n > N \Rightarrow 0 < |s_n - c| < \delta \Rightarrow |f(s_n) - L| < \epsilon,$$

which implies that $\lim_{n\to\infty} f(s_n) = L$.

Conversely, suppose that $\lim_{x\to c} f(x) \neq L$. Then, there exists a $\epsilon > 0$ such that for all $\delta > 0$ there exists an $x \in S$ such that $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$. In particular, for each $n \in \mathbb{N}$ there exists a $s_n \in S$ such that $0 < |s_n - c| < 1/n$ and $|f(s_n) - L| \geq \epsilon$. Therefore, $\lim_{n\to\infty} s_n = c$ and $\lim_{n\to\infty} f(s_n) \neq L$. □

Using Theorem 2.4, we can start applying everything we know about sequential limits to limits of functions, e.g., see the Limit Theorems from October 2, 2023.

Theorem 2.2. *Let $f\colon S \to \mathbb{R}$, $g\colon S \to \mathbb{R}$, and c be an accumulation point of S. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = L'$, then the following hold*

a. $\lim_{x\to c} (f(x) + g(x)) = L + L'$,

b. $\lim_{x\to c} (kf(x)) = kL$, *for all* $k \in \mathbb{R}$,

c. $\lim_{x\to c} (f(x) \cdot g(x)) = L \cdot L'$,

d. $\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{L'}$, *if* $L' \neq 0$.

Theorem 2.3. *Let $f\colon S \to \mathbb{R}$, $g\colon S \to \mathbb{R}$, and c be an accumulation point of S. Suppose that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = L'$. If $f(x) \leq g(x)$ for all $x \in S \setminus \{c\}$, then $L \leq L'$.*

2.1 Continuous Functions

Definition 2.4. Let $f\colon S \to \mathbb{R}$ and let $c \in S$. Then, f is *continuous at* c if

$$\forall \epsilon > 0,\ \exists \delta > 0 \ \ni \ |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$$

If f is continuous at each point $c \in S$, then we say that f is *continuous on* S. If f is continuous on its domain S, then we say that f is *continuous*.

Proposition 2.5. *Let $f\colon S \to \mathbb{R}$ and let $c \in S$. If c is not an accumulation point of S, then f is continuous at c.*

Proof. Let $\epsilon > 0$. Since c is an isolated point of S, there exists a $\delta > 0$ such that $N(c;\delta) \cap S = \{c\}$. Therefore,

$$|x - c| < \delta \Rightarrow x = c \Rightarrow |f(x) - f(c)| = 0 < \epsilon.$$

□

Theorem 2.6. *Let $f\colon S \to \mathbb{R}$ and let $c \in S$. Suppose c is an accumulation point of S. Then, f is continuous at c if and only if*

$$\lim_{x \to c} f(x) = f(c).$$

Proof. Suppose f is continuous at c and let $\epsilon > 0$. Then, there exists a $\delta > 0$ such that

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

Therefore,

$$0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

So, $\lim_{x\to c} f(x) = f(c)$.

Suppose that $\lim_{x\to c} f(x) = f(c)$ and let $\epsilon > 0$. Then, there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

If $|x - c| = 0$, then $x = c$ so $|f(x) - f(c)| = 0 < \epsilon$. Therefore,

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

So, f is continuous at c. □

3 Exercises

I. Prove Proposition 2.1

II. Prove Theorem 2.4.

III. Use Theorem 2.4 to show that $f(x) = \sin(1/x)$ does not converge as x approaches 0.

IV. Use Theorem 2.4 to show that $\lim_{x\to 0} x \sin(1/x)$ does converge as x approaches 0.

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.