Real Analysis

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1 Daily Quiz

Let $f\colon S \to \mathbb{R}$ and let $c \in S$. State the definition of f being continuous at c.

2 Key Topics

Today we finish our discussion of continuous functions. For further reading, see [1, Sections 3.2–3.3]. Note that [1] refers to an accumulation point as a cluster point.

We begin with the following theorem.

Theorem 2.1. *Let $f\colon S \to \mathbb{R}$ and let $c \in S$. Suppose c is an accumulation point of S. Then, f is continuous at c if and only if*

$$\lim_{x\to c} f(x) = f(c).$$

Proof. Suppose f is continuous at c and let $\epsilon > 0$. Then, there exists a $\delta > 0$ such that

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

Therefore,

$$0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

So, $\lim_{x\to c} f(x) = f(c)$.

Suppose that $\lim_{x\to c} f(x) = f(c)$ and let $\epsilon > 0$. Then, there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

If $|x - c| = 0$, then $x = c$ so $|f(x) - f(c)| = 0 < \epsilon$. Therefore,

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

So, f is continuous at c. □

We give an equivalent definition of continuity in terms of sequential limits in the following theorem.

Theorem 2.2. *Let $f\colon S \to \mathbb{R}$ and let $c \in S$. Then, f is continuous at c if and only if for all $s\colon \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} s_n = c$, we have*

$$\lim_{n\to\infty} f(s_n) = f(c).$$

2.1 Properties of Continuous Functions

Proposition 2.3. *Let $f\colon B \to \mathbb{R}$ and $g\colon A \to B$. If g is continuous at $c \in A$ and f is continuous at $g(c)$. Then, $f \circ g\colon A \to \mathbb{R}$ is continuous at c.*

Proof. Suppose that $x\colon \mathbb{N} \to A$ satisfies $\lim_{n\to\infty} x_n = c$. Then, by continuity of g at c, $g(s)\colon \mathbb{N} \to B$ satisfies $\lim_{n\to\infty} g(s_n) = g(c)$. Finally, by continuity of f at $g(c)$, $f(g(s))\colon \mathbb{N} \to \mathbb{R}$ satisfies

$$\lim_{n\to\infty} f(g(s_n)) = f(g(c)).$$

Therefore, $f \circ g$ is continuous at c. □

Lemma 2.4. *Let $S \subseteq \mathbb{R}$ be compact (closed and bounded). Then, any continuous function $f\colon S \to \mathbb{R}$ is bounded.*

Proof. Suppose $f\colon S \to \mathbb{R}$ is not bounded. Then, for each $n \in \mathbb{N}$, there is a $s_n \in S$ such that

$$|f(s_n)| \geq n.$$

The range $\operatorname{rng}(s) \subseteq S$ is bounded. Hence, the Bolzano-Weirstrass theorem implies that $\operatorname{rng}(s)$ has an accumulation point, $a \in S$. Note that, since S is compact (closed and bounded), we know that S contains all of its accumulation points.

Since a is an accumulation point of $\operatorname{rng}(s)$, it follows that there exists a subsequence s_{n_k} such that $\lim_{k\to\infty} s_{n_k} = a$. However, $|f(s_{n_k})| \geq n_k \geq k$, so $\lim_{k\to\infty} f(s_{n_k})$ does not exist. Therefore, f is not continuous at $a \in S$. □

Theorem 2.5. *Let $S \subseteq \mathbb{R}$ be compact (closed and bounded) and let $f\colon S \to \mathbb{R}$ be continuous. Then, $f(S) \subseteq \mathbb{R}$ is compact.*

Proof. Note that Lemma 2.4 implies that $f(S)$ is bounded. If $f(S)'$ is empty, then we are done since $f(S)$ is bounded and closed since it contains all of its accumulation points. Suppose $f(S)'$ is non-empty and let $b \in f(S)'$. Then, for each $n \in \mathbb{N}$, there exists a $y_n \in N^*(b; 1/n) \cap F(S)$ and a $x_n \in S$ such that $f(x_n) = y_n$. Note that the y_n can be selected to be distinct; hence, $\operatorname{rng}(y)$ and $\operatorname{rng}(x)$ are infinite sets. Since $\operatorname{rng}(x) \subseteq S$ is bounded, the Bolzano-Weirstrass theorem implies that $\operatorname{rng}(x)$ has an accumulation point, $a \in S$. Therefore, there exists a subsequence x_{n_k} such that

$$\lim_{k\to\infty} x_{n_k} = a.$$

Since f is continuous, we have

$$f(a) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = b.$$

Since $f(a) = b$, it follows that $b \in f(S)$.

□

Corollary 2.6. *Let $S \subseteq \mathbb{R}$ be compact (closed and bounded) and let $f\colon S \to \mathbb{R}$ be continuous. Then, f achieves both its maximum and minimum values on S.*

Proof. Since $f(S) \subseteq \mathbb{R}$ is bounded, the completeness axiom implies that $f(S)$ has an infimum and supremum. Furthermore, since $f(S)$ is closed, both the infimum and supremum are elements of $f(S)$. Hence, there exists $s_1, s_2 \in S$ such that

$$f(s_1) = \inf f(S) \text{ and } f(s_2) = \sup f(S).$$

□

3 Exercises

I. Prove Theorem 2.2.

II. Prove Lemma 2.4.

III. Prove Theorem 2.5.

IV. Prove Corollary 2.6.

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.