Real Analysis

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1 Daily Quiz

Let $f\colon S \to \mathbb{R}$ and let $c \in S$. State the definition of f being continuous at c.

2 Key Topics

Today we finish proving that the image of a compact set under a continuous function is compact. Then, we introduce the concept of uniform continuity. For further reading, see [1, Sections 3.3–3.4]. Note that [1] refers to an accumulation point as a cluster point.

2.1 Properties of Continuous Functions

On October 16, 2023, we proved the following lemma.

Lemma 2.1. *Let $S \subseteq \mathbb{R}$ be compact (closed and bounded). Then, any continuous function $f\colon S \to \mathbb{R}$ is bounded.*

Today, we use this lemma to prove our main result.

Theorem 2.2. *Let $S \subseteq \mathbb{R}$ be compact (closed and bounded) and let $f\colon S \to \mathbb{R}$ be continuous. Then, $f(S) \subseteq \mathbb{R}$ is compact.*

Proof. Note that Lemma 2.1 implies that $f(S)$ is bounded. If $f(S)'$ is empty, then we are done since $f(S)$ is bounded and closed since it contains all of its accumulation points. Suppose $f(S)'$ is non-empty and let $b \in f(S)'$. Then, for each $n \in \mathbb{N}$, there exists a $y_n \in N^*(b; 1/n) \cap F(S)$ and a $x_n \in S$ such that $f(x_n) = y_n$. Note that the y_n can be selected to be distinct; hence, $\operatorname{rng}(y)$ and $\operatorname{rng}(x)$ are infinite sets. Since $\operatorname{rng}(x) \subseteq S$ is bounded, the Bolzano-Weirstrass theorem implies that $\operatorname{rng}(x)$ has an accumulation point, $a \in S$. Therefore, there exists a subsequence x_{n_k} such that

$$\lim_{k\to\infty} x_{n_k} = a.$$

Since f is continuous, we have

$$f(a) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = b.$$

Since $f(a) = b$, it follows that $b \in f(S)$.

Therefore, $f(S)$ contains all of its accumulation points, so $f(S)$ is compact (closed and bounded). □

As a consequence of our main result, it follows that continuous functions over bounded domains attain their maximum and minimum values.

Corollary 2.3. *Let $S \subseteq \mathbb{R}$ be compact (closed and bounded) and let $f\colon S \to \mathbb{R}$ be continuous. Then, f achieves both its maximum and minimum values on S.*

Proof. Since $f(S) \subseteq \mathbb{R}$ is bounded, the completeness axiom implies that $f(S)$ has an infimum and supremum. Furthermore, since $f(S)$ is closed, both the infimum and supremum are elements of $f(S)$. Hence, there exists $s_1, s_2 \in S$ such that

$$f(s_1) = \inf f(S) \text{ and } f(s_2) = \sup f(S).$$

□

2.2 Uniform Continuity

Definition 2.4. Let $S \subseteq \mathbb{R}$ and let $f\colon S \to \mathbb{R}$. We say that f is *uniformly continuous on S* if

$$\forall \epsilon > 0,\ \exists \delta > 0\ \ni\ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Example 2.5. We will show that $f(x) = 2x$ is uniformly continuous on $\mathbb{R}$. Let $\epsilon > 0$ and define $\delta = \epsilon/2$. Then,

$$\begin{aligned} |x-y| < \delta &\Rightarrow |x-y| < \frac{\epsilon}{2} \\ &\Rightarrow |f(x) - f(y)| = 2\,|x-y| < \epsilon. \end{aligned}$$

Example 2.6. We will show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. Let $\epsilon = 1$ and $\delta > 0$. Then, define $x = 1/\delta$ and $y = 1/\delta + \delta/2$, which gives us

$$|x-y| = \frac{\delta}{2} < \delta$$

and

$$\begin{aligned} |f(x) - f(y)| &= |x+y|\,|x-y| \\ &= \left|\frac{2}{\delta} + \frac{\delta}{2}\right| |\delta|\,2 \\ &> \frac{2}{\delta}\frac{\delta}{2} = 1 = \epsilon. \end{aligned}$$

The issue in the previous example was that $|x+y|$ could be made arbitrarily large, this could not happen over a bounded interval.

Example 2.7. We will show that $f(x) = 1/x$ is not uniformly continuous on $(0,2)$. Let $\epsilon = 1$ and $\delta > 0$. Define $x_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} x_n = 0$. Hence, x is Cauchy since it is convergent. Therefore, there is a N such that $n, m > N$ implies that

$$|x_n - x_m| < \delta.$$

Fix $m > N$ and let $n = m + 2$. Then, $|x_n - x_m| < \delta$ but

$$|f(x_n) - f(x_m)| = 2 > \epsilon.$$

The issue in the previous example was that $f(x)$ is unbounded as $x \to 0$. Since continuity over a compact interval implies that the range is bounded, this would not happen if the domain of f were compact.

3 Exercises

I. Prove Theorem 2.2.

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.