Real Analysis

Thomas R. Cameron

October 2, 2023

1 Daily Quiz

Show that $s_n = 1 - \frac{1}{n}$ converges to 1.

2 Key Topics

Having established the limit of a sequence:

$$\lim_{n\to\infty} s_n = L,$$

if $\forall \epsilon > 0,\ \exists N \in \mathbb{R} \ni n > N \Rightarrow |s_n - L| < \epsilon$, we now establish basic arithmetic properties of the limit. For further reading, see [1, Section 2.2].

Today, we will prove the following theorem.

Theorem 2.1. *Let $s\colon \mathbb{N} \to \mathbb{R}$ and $t\colon \mathbb{N} \to \mathbb{R}$ be convergent with limits L and L', respectively. Then, the following hold*

a. $\lim_{n\to\infty}(s_n + t_n) = L + L'$,

b. $\lim_{n\to\infty}(ks_n) = kL$, *for any* $k \in \mathbb{R}$,

c. $\lim_{n\to\infty}(s_n \cdot t_n) = L \cdot L'$,

d. $\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right) = \frac{L}{L'}$, *if* $L' \neq 0$.

Proof.

a. Let $\epsilon > 0$. Then, there exists $N_1, N_2 \in \mathbb{R}$ such that

$$\begin{aligned} n > N_1 &\Rightarrow |s_n - L| < \frac{\epsilon}{2} \\ n > N_2 &\Rightarrow |t_n - L'| < \frac{\epsilon}{2}. \end{aligned}$$

Then, let $N = \max\{N_1, N_2\}$ so that

$$\begin{aligned} n > N \Rightarrow |(s_n + t_n) - (L + L')| &= |(s_n - L) + (t_n - L')| \\ &\leq |s_n - L| + |t_n - L'| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

b. Let $\epsilon > 0$. Then, there exists $N \in \mathbb{R}$ such that

$$n > N \Rightarrow |s_n - L| < \frac{\epsilon}{|k| + 1}.$$

Then, we have

$$\begin{aligned} n > N \Rightarrow |ks_n - kL| &= |k|\,|s_n - L| \\ &< (|k|+1)\,|s_n - L| \\ &< (|k|+1)\,\frac{\epsilon}{|k|+1} = \epsilon. \end{aligned}$$

c. Let $\epsilon > 0$. Then, there exists $N_1, N_2 \in \mathbb{R}$ such that

$$\begin{aligned} n > N_1 &\Rightarrow |s_n - L| < \frac{\epsilon}{2(M+1)} \\ n > N_2 &\Rightarrow |t_n - L'| < \frac{\epsilon}{2(|L|+1)}, \end{aligned}$$

where M is an upper bound on $|t_n|$, which is guaranteed to exist since t is a convergent sequence. Then, let $N = \max\{N_1, N_2\}$ so that

$$\begin{aligned} n > N \Rightarrow |s_n t_n - LL'| &= |s_n t_n - t_n L + t_n L - LL'| \\ &\leq |t_n|\,|s_n - L| + |L|\,|t_n - L'| \\ &\leq (M+1)\,|s_n - L| + (|L|+1)\,|t_n - L'| \\ &< (M+1)\frac{\epsilon}{2(M+1)} + (|L|+1)\frac{\epsilon}{2(|L|+1)} = \epsilon. \end{aligned}$$

d. Let $\epsilon > 0$. Then, there exists $N_1, N_2 \in \mathbb{R}$ such that

$$\begin{aligned} n > N_1 &\Rightarrow |s_n - L| < \frac{|L'|\,\epsilon}{4} \\ n > N_2 &\Rightarrow |t_n - L'| < \frac{|L'|^2\,\epsilon}{4(|L|+1)} \end{aligned}$$

Note that, since t converges to L', it follows that there is a $N_3 \in \mathbb{R}$ such that

$$n > N_3 \Rightarrow |t_n| \geq \frac{|L'|}{2}.$$

Then, let $N = \max\{N_1, N_2, N_3\}$ so that

$$\begin{aligned} n > N \Rightarrow \left|\frac{s_n}{t_n} - \frac{L}{L'}\right| &= \left|\frac{L's_n - Lt_n}{L't_n}\right| \\ &= \frac{1}{|L't_n|}\,|L's_n - Lt_n| \\ &\leq \frac{2}{|L'|^2}\,|L's_n - Lt_n| \\ &= \frac{2}{|L'|^2}\,|L's_n - LL' + LL' - Lt_n| \\ &\leq \frac{2}{|L'|^2}\,(|L'|\,|s_n - L| + |L|\,|t_n - L'|) \\ &\leq \frac{2}{|L'|}\,|s_n - L| + 2\frac{(|L|+1)}{|L'|^2}\,|t_n - L'| \\ &< \frac{2}{|L'|}\frac{|L'|\,\epsilon}{4} + 2\frac{(|L|+1)}{|L'|^2}\frac{|L'|^2\,\epsilon}{4(|L|+1)} = \epsilon. \end{aligned}$$

□

In addition, we will prove the following results.

Theorem 2.2. *Let* $s\colon \mathbb{N} \to \mathbb{R}$ *and* $t\colon \mathbb{N} \to \mathbb{R}$ *be convergent with limits* L *and* L'*, respectively. If* $s_n \le t_n$ *for all* $n \in \mathbb{N}$*, then* $L \le L'$*.*

Proof. For the sake of contradiction, suppose that $L > L'$. Define $\epsilon = \frac{L-L'}{2} > 0$. Then, there exists $N_1, N_2 \in \mathbb{R}$ such that

$$n > N_1 \Rightarrow L - \epsilon < s_n < L + \epsilon$$
$$n > N_2 \Rightarrow L' - \epsilon < t_n < L' + \epsilon.$$

Then, let $N = \max\{N_1, N_2\}$ so that

$$n > N \Rightarrow t_n < L' + \epsilon = \frac{L + L'}{2} = L - \epsilon < s_n,$$

which contradicts $s_n \le t_n$ for all $n \in \mathbb{N}$. □

Theorem 2.3. *Let* $s\colon \mathbb{N} \to \mathbb{R}$ *with* $s_n > 0$ *for all* $n \in \mathbb{N}$*. If*

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = L < 1,$$

then $\lim_{n\to\infty} s_n = 0$*.*

Proof. There exists a $c \in \mathbb{R}$ such that $L < c < 1$. Define $\epsilon = (c - L) > 0$. Then, there is a $N \in \mathbb{R}$ such that

$$n > N \Rightarrow \frac{s_{n+1}}{s_n} < \epsilon + L = c.$$

Therefore, for $n > N$, we have

$$0 < s_{n+1} < cs_n < c^2 s_{n-1} < \cdots < c^{n-N+1} s_N.$$

Define $M = \frac{s_N}{c^N}$. Then, for all $n > N$, we have

$$s_{n+1} < Mc^{n+1}.$$

Since $\lim_{n\to\infty} c^{n+1} = 0$, it follows that $\lim_{n\to\infty} s_{n+1} = 0$. □

3 Exercises

I. Prove Theorem 2.1

II. Prove Theorem 2.2

III. Prove Theorem 2.3

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.