Real Analysis

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1 Daily Quiz

Let $f\colon S \to \mathbb{R}$. State the definition of f being uniformly continuous on S.

2 Key Topics

Today we introduce the definition of the derivative. For further reading, see [1, Section 4.1].

2.1 The Derivative

Definition 2.1. Let I be an interval and let $f\colon I \to \mathbb{R}$. Then, f is *differentiable* at $c \in I$ if there exists an $L \in \mathbb{R}$ such that

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = L$$

In this case, we say that L is the derivative of f at c and we write $f'(c) = L$. If f is differentiable at all $c \in I$, then we say that f is differentiable on I and we defined the derivative function $f'\colon I \to \mathbb{R}$ such that

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c},$$

for all $c \in I$.

Example 2.2. We will show that $f(x) = x^2$ is differentiable on $\mathbb{R}$. To this end, let $c \in \mathbb{R}$ and consider the limit

$$\begin{aligned}\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to c}\frac{x^2-c^2}{x-c}\\ &= \lim_{x\to c}\frac{(x+c)(x-c)}{x-c}\\ &= \lim_{x\to c}(x+c) = 2c.\end{aligned}$$

Therefore, the derivative of $f(x) = x^2$ is give by $f'(x) = 2x$.

2.2 Sequential Limit

Using the sequential criterion for limits, see Theorem 2.4 from October 11 2023, we can arrive at the following theorem.

Theorem 2.3. *Let I be an interval and let $f\colon I \to \mathbb{R}$. Then, f is* differentiable *at $c \in I$ if and only if there exists an $L \in \mathbb{R}$ such that for every sequence $s\colon \mathbb{N} \to I$, where $\operatorname{rng}(s) \subseteq I \setminus \{c\}$ and $\lim_{n\to\infty} s_n = c$, we have*

$$\lim_{n\to\infty}\frac{f(s_n)-f(c)}{s_n-c} = L.$$

Example 2.4. We will show that $f(x) = |x|$ is not differentiable at $c = 0$. To this end, let $s\colon \mathbb{N} \to \mathbb{R}$ be defined by

$$s_n = \frac{(-1)^n}{n}.$$

Clearly, $\lim_{n\to\infty} s_n = 0$. However,

$$\begin{aligned}\lim_{n\to\infty} \frac{f(s_n) - f(0)}{s_n - 0} &= \lim_{n\to\infty} \frac{|s_n|}{s_n} \\ &= \lim_{n\to\infty} (-1)^n,\end{aligned}$$

which does not converge.

Finally, we will show that differentiability implies continuity.

Theorem 2.5. *Let I be an interval and let $f\colon I \to \mathbb{R}$. If f is differentiable at $c \in I$, then f is continuous at c.*

Proof. Suppose that f is differentiable at $c \in I$. Since I is an interval, every $c \in I$ is an accumulation point. Hence, we can show that f is continuous at c by showing that $\lim_{x\to c} f(x) = f(c)$.

To this end, let $s\colon \mathbb{N} \to I$, where $\operatorname{rng}(s) \subseteq I \setminus \{c\}$ and $\lim_{n\to\infty} s_n = c$. Then, there exists an $L \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{f(s_n) - f(c)}{s_n - c} = L.$$

Therefore, we have

$$\begin{aligned}\lim_{n\to\infty} f(s_n) &= \lim_{n\to\infty} \left((s_n - c)\frac{f(s_n) - f(c)}{s_n - c} + f(c) \right) \\ &= \lim_{n\to\infty} (s_n - c) \cdot \lim_{n\to\infty} \frac{f(s_n) - f(c)}{s_n - c} + f(c) \\ &= 0 \cdot L + f(c) = f(c).\end{aligned}$$

□

3 Exercises

I. Prove Theorem 2.5.

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.