Real Analysis

Thomas R. Cameron

October 25, 2023

1 Daily Quiz

2 Key Topics

Today we derive the basic rules of differentiation. For further reading, see [1, Section 4.1].

Recall the definition of the derivative.

Definition 2.1. Let I be an interval and let $f\colon I \to \mathbb{R}$. Then, f is *differentiable* at $c \in I$ if there exists an $L \in \mathbb{R}$ such that

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = L$$

In this case, we say that L is the derivative of f at c and we write $f'(c) = L$. If f is differentiable at all $c \in I$, then we say that f is differentiable on I and we defined the derivative function $f'\colon I \to \mathbb{R}$ such that

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c},$$

for all $c \in I$.

Example 2.2. Suppose that $f\colon I \to \mathbb{R}$ is differentiable at $c \in I$. We will show that $(kf)'(c) = kf'(c)$, for any $k \in \mathbb{R}$. Note that this is the constant multiple rule of differentiation.

To this end, note that

$$\lim_{x\to c} \frac{kf(x)-kf(c)}{x-c} = k\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = kf'(c).$$

Example 2.3. Suppose that $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ are differentiable at $c \in I$. We will show that $(f+g)'(c) = f'(c) + g'(c)$. Note that this is the sum rule of differentiation.

To this end, note that

$$\begin{aligned}
\lim_{x\to c} \frac{(f+g)(x)-(f+g)(c)}{x-c} &= \lim_{x\to c} \frac{f(x)+g(x)-f(c)-g(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\
&= f'(c) + g'(c).
\end{aligned}$$

3 Exercises

Suppose that $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ are differentiable at $c \in I$.

I. Prove the product rule of differentiation:

$$(fg)'(c) = f(c)g'(c) + g(c)f'(c).$$

Hint: Add $(-f(c)g(x) + f(c)g(x))$, i.e., zero, to the numerator of $\lim_{x\to c} \frac{(fg)(x)-(fg)(c)}{x-c}$.

II. Prove the quotient rule of differentiation:

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g(c)^2},$$

assuming $g(c) \neq 0$.

Hint: Apply the following steps to $\lim_{x\to c} \frac{(f/g)(x)-(f/g)(c)}{x-c}$:

- Find a common denominator and write $(f/g)(x) - (f/g)(c)$ as a single fraction.
- Add $(-g(c)f(c) + g(c)f(c))$ to the numerator of the single fraction from the previous step.

III. Prove the power rule:

$$\frac{d}{dx}x^n = nx^{n-1},\ \forall n \in \mathbb{N}.$$

Hint: Prove by mathematical induction, use the product rule on the induction step.

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.