Real Analysis

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1 Daily Quiz

2 Key Topics

Today we prove the mean value theorem. For further reading, see [1, Section 4.2].

We begin with a definition of relative extrema.

Definition 2.1. Let $S \subseteq \mathbb{R}$ and $f\colon S \to \mathbb{R}$. Then, $c \in S$ is a *relative max* if there exists an $\delta > 0$ such that

$$\forall x \in N(c;\delta),\ f(x) \leq f(c).$$

Similarly, $c \in S$ is a *relative min* if there exists an $\delta > 0$ such that

$$\forall x \in N(c;\delta),\ f(x) \geq f(c).$$

If c is either a relative max or a relative min, then we say that c is a *relative extrema* of f.

3 Exercises

I. Prove the following Lemma.

Lemma 3.1. *Suppose that $f\colon (a,b) \to \mathbb{R}$ is differentiable on (a,b). If $c \in (a,b)$ is a relative extrema of f, then $f'(c) = 0$.*

Hint: Apply the following steps:

- Assume that c is a relative max, note that if c is a relative min of f then c is a relative max of $-f$.
- Fix $\delta > 0$ such that $\forall x \in N(c;\delta),\ f(x) \leq f(c)$.
- Let $x\colon \mathbb{N} \to (c-\delta, c)$ such that $\lim_{n\to\infty} y_n = c$. Explain why
$$\frac{f(x_n) - f(c)}{x_n - c} \geq 0.$$
- Let $y\colon \mathbb{N} \to (c, c+\delta)$ such that $\lim_{n\to\infty} y_n = c$. Explain why
$$\frac{f(y_n) - f(c)}{y_n - c} \leq 0.$$
- Explain why the previous two steps imply that $f'(c) = 0$.

II. Prove Rolle's theorem.

Theorem 3.2. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then there is a $c \in (a,b)$ such that $f'(c) = 0$.*

Hint: Apply the following steps:

- Use Corollary 2.6 to explain why there exists an $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.
- Break the rest of the argument into two cases: Case 1 is where x_1 and x_2 are both end points, Case 2 is where either $x_1 \in (a, b)$ or $x_2 \in (a, b)$.
- In Case 1, explain why f is a constant function and therefore $f'(c) = 0$ for all $c \in (a, b)$.
- In Case 2, explain why either x_1 or x_2 is a relative extrema and use Lemma 3.1 to conclude that $f'(x_1) = 0$ or $f'(x_2) = 0$.

III. Prove the Mean Value Theorem.

Theorem 3.3. *Suppose that* $f \colon [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$ *and differentiable on* (a, b). *Then, there exists a* $c \in (a, b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Hint: Apply the following steps:

- Define
$$g(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a), \ \forall x \in [a, b].$$
- Explain why you can apply Rolle's theorem to $h(x) = f(x) - g(x)$.
- Conclude that there is a $c \in (a, b)$ such that $h'(c) = f'(c) - g'(c) = 0$.

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.