Real Analysis

Thomas R. Cameron

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1 Daily Quiz

State the Mean Value Theorem

2 Key Topics

Today we review our proof of the Mean Value Theorem from Homework 5.2. In addition, we use the Mean Value Theorem to prove the so-called "first derivative test". For further reading, see [1, Section 4.2].

2.1 Mean Value Theorem

Lemma 2.1. *Suppose that $f\colon (a,b) \to \mathbb{R}$ is differentiable on (a,b). If $c \in (a,b)$ is a relative extrema of f, then $f'(c) = 0$.*

Proof. Without loss of generality, suppose c is a relative max. Then, there exists a $\delta > 0$ such that for all $x \in N(c;\delta)$, $f(x) \leq f(c)$. Let $x\colon \mathbb{N} \to (c-\delta, c)$ such that $\lim_{n\to\infty} x_n = c$. Then,

$$\frac{f(x_n) - f(c)}{x_n - c} \geq 0,$$

for all $n \in \mathbb{N}$. Therefore, $\lim_{n\to\infty} \frac{f(x_n)-f(c)}{x_n - c} \geq 0$.

Also, let $y\colon \mathbb{N} \to (c, c+\delta)$ such that $\lim_{n\to\infty} y_n = c$. Then,

$$\frac{f(y_n) - f(c)}{y_n - c} \leq 0,$$

for all $n \in \mathbb{N}$. Therefore, $\lim_{n\to\infty} \frac{f(y_n)-f(c)}{y_n - c} \leq 0$.

Since f is differentiable at c, it follows that $f'(c) = 0$. □

Theorem 2.2 (Rolle's). *Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then there is a $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. Since f is continuous and $[a,b]$ is compact, it follows that $f([a,b])$ is compact and therefore contains its infimum and supremum values. Therefore, there exists an $x_1, x_2 \in [a,b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a,b]$.

If x_1 and x_2 are both endpoints, then $f(x) =$ is constant on $[a,b]$ so $f'(c) = 0$ for all $c \in (a,b)$.

If either x_1 or x_2 are not endpoints, then they are relative extrema and it follows that $f'(x_1) = 0$ or $f'(x_2) = 0$. □

Theorem 2.3. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then, there exists a $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define

$$g(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a),$$

for all $x \in [a, b]$. Then, g is continuous on $[a, b]$ and differentiable on (a, b). Furthermore, $h(x) = f(x) - g(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and $h(a) = 0$ and $h(b) = 0$. Thus, Rolle's theorem applied to $h(x)$ gives us a $c \in (a, b)$ such that $h'(c) = 0$. Note that $h'(c) = f'(c) - g'(c)$. Therefore, we have

$$f'(c) = g'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

2.2 First Derivative Test

Proposition 2.4.

Let I be an open interval and let $f\colon I \to \mathbb{R}$ be differentiable. Then,

- *I. If $f'(x) \geq 0$ for all $x \in I$, then f is monotone increasing.*
- *II. If $f'(x) \leq 0$ for all $x \in I$, then f is monotone decreasing.*

Proof. I. Suppose that $f'(x) \geq 0$ for all $x \in I$ and let $x_1 < x_2$ be elements of I. Then, the MVT applied to $f\colon [x_1, x_2] \to \mathbb{R}$ gives us a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Therefore, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \geq 0$, so $f(x_2) \geq f(x_1)$.

II. Similar to part I.

□

Proposition 2.5. *Let $f\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Also, let $c \in (a, b)$.*

- *I. If there is a $\delta > 0$ such that $f'(x) \leq 0$ for all $x \in (c - \delta, c)$ and $f'(x) \geq 0$ for all $x \in (c, c + \delta)$, then c is a relative min.*
- *II. If there is a $\delta > 0$ such that $f'(x) \geq 0$ for all $x \in (c - \delta, c)$ and $f'(x) \leq 0$ for all $x \in (c, c + \delta)$, then c is a relative max.*

3 Exercises

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.