Real Analysis

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October 4, 2023

1 Daily Quiz

Use the algebraic properties of the limit to find

$$\lim_{n\to\infty}\left(\frac{3n^2-5n}{4n^2+3n+2}\right)$$

2 Key Topics

Having established the definition of a convergent sequence, we now investigate monotone and Cauchy sequences For further reading, see [1, Sections 2.1.1 and 2.4].

We begin by completing the following results.

Theorem 2.1. *Let $s\colon \mathbb{N}\to\mathbb{R}$ and $t\colon\mathbb{N}\to\mathbb{R}$ be convergent with limits L and L', respectively. If $s_n\leq t_n$ for all $n\in\mathbb{N}$, then $L\leq L'$.*

Proof. For the sake of contradiction, suppose that $L>L'$. Define $\epsilon=\frac{L-L'}{2}>0$. Then, there exists $N_1,N_2\in\mathbb{R}$ such that

$$\begin{aligned} n>N_1 &\Rightarrow L-\epsilon<s_n<L+\epsilon\\ n>N_2 &\Rightarrow L'-\epsilon<t_n<L'+\epsilon. \end{aligned}$$

Then, let $N=\max\{N_1,N_2\}$ so that

$$n>N\Rightarrow t_n<L'+\epsilon=\frac{L+L'}{2}=L-\epsilon<s_n,$$

which contradicts $s_n\leq t_n$ for all $n\in\mathbb{N}$. □

Theorem 2.2. *Let $s\colon\mathbb{N}\to\mathbb{R}$ with $s_n>0$ for all $n\in\mathbb{N}$. If*

$$\lim_{n\to\infty}\frac{s_{n+1}}{s_n}=L<1,$$

then $\lim_{n\to\infty}s_n=0$.

Proof. There exists a $c\in\mathbb{R}$ such that $L<c<1$. Define $\epsilon=(c-L)>0$. Then, there is a $N\in\mathbb{R}$ such that

$$n>N\Rightarrow\frac{s_{n+1}}{s_n}<\epsilon+L=c.$$

Therefore, for $n>N$, we have

$$0<s_{n+1}<cs_n<c^2s_{n-1}<\cdots<c^{n-N+1}s_N.$$

Define $M=\frac{s_N}{c^N}$. Then, for all $n>N$, we have

$$s_{n+1}<Mc^{n+1}.$$

Since $\lim_{n\to\infty}c^{n+1}=0$, it follows that $\lim_{n\to\infty}s_{n+1}=0$. □

2.1 Monotone Sequences

Definition 2.3. A sequence $s\colon \mathbb{N} \to \mathbb{R}$ is *increasing* if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$. Similarly, s is *decreasing* if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is increasing or decreasing.

Theorem 2.4. *A monotone sequence converges if and only if it is bounded.*

2.2 Cauchy Sequences

Definition 2.5. A sequence $s\colon \mathbb{N} \to \mathbb{R}$ is a *Cauchy sequence* if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{R} \ \ni \ m, n > N \ \Rightarrow \ |s_n - s_m| < \epsilon.$$

Lemma 2.6. *Every Cauchy sequence is bounded.*

Theorem 2.7. *A sequence $s\colon \mathbb{N} \to \mathbb{R}$ is convergent if and only if it is a Cauchy sequence.*

3 Exercises

I. Prove Theorem 2.1

II. Prove Theorem 2.2

III. Prove Theorem 2.4

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.