Real Analysis

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1 Daily Quiz

Show that the following sequence is not Cauchy:

$$s_n = 3n.$$

2 Key Topics

Today we prove that every real Cauchy sequence is convergent in $\mathbb{R}$. For further reading, see [**?**, 2.4].

We begin with the following lemma which establishes that Cauchy sequences are bounded.

Lemma 2.1. *Every Cauchy sequence is bounded.*

Proof. Let $s\colon \mathbb{N} \to \mathbb{R}$ be Cauchy and define $\epsilon = 1$. Then, there exists an $N \in \mathbb{R}$ such that $n, m > N \Rightarrow |s_n - s_m| < 1$. Fix m as the smallest integer bigger than N. Then, we have

$$n > N \Rightarrow |s_n - s_m| < 1 \Rightarrow |s_n| < 1 + |s_m|.$$

Then, the following serves as an upper bound for s:

$$M = \max\{|s_1|, \ldots, |s_m|, 1 + |s_m|\}.$$

□

Next, we prove our main result.

Theorem 2.2. *A real sequence is convergent if and only if it is Cauchy.*

Proof. On October 4, we proved that every convergent sequence is Cauchy.

Let $s\colon \mathbb{N} \to \mathbb{R}$ be Cauchy and define its range to be

$$R = \{s_n\colon\ n \in \mathbb{N}\}.$$

We break the remainder of this proof into two cases: R is finite and R is infinite.

If R is finite, then there exists an $\epsilon > 0$ such that $N(x;\epsilon) \cap R = \{x\}$ for all $x \in R$. Since s is Cauchy, there exists an N such that $n, m > N \Rightarrow |s_n - s_m| < \epsilon$. Fix m as the smallest integer bigger than N. Then, we have

$$n > N \Rightarrow |s_n - s_m| < \epsilon \Rightarrow s_n = s_m.$$

Therefore, $\lim_{n\to\infty} s_n = s_m$.

If R is infinite, then Lemma 2.1 and the Bolzano-Weirstrass Theorem (see Review 1) implies that R has an accumulation point, which we denote by $a \in \mathbb{R}$. Let $\epsilon > 0$. Then, there is an $N \in \mathbb{R}$ such that

$$n, m > N \Rightarrow |s_n - s_m| < \frac{\epsilon}{2}.$$

Since a is an accumulation point, $N(a;\epsilon/2)$ contains infinitely many points of R. Hence, there is an $m > N$ such that $s_m \in N(a;\epsilon/2)$. Then, for all $n > N$, we have

$$\begin{aligned} |s_n - a| &= |s_n - s_m + s_m - a| \\ &\leq |s_n - s_m| + |s_m - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

3 Group Work

In class, we proved Theorem 2.2 in 3 groups.

3.1 Group 1

Prove that if R is finite, then s converges to an element of the sequence.

Hint: There is an $\epsilon > 0$ such that $N(x;\epsilon) \cap R = \{x\}$ for all $x \in R$.

3.2 Group 2

Prove that if R is infinite, then there is an accumulation point of R in $\mathbb{R}$.

Hint: Reference review 1.

3.3 Group 3

Prove that if R is infinite, then $\lim_{n\to\infty} s_n = a$, where $a \in \mathbb{R}$ is an accumulation point of R.

Hint: You can assume that R has an accumulation point. Since a is an accumulation point, every neighborhood of a contains infinitely many points of R. Finally, note that

$$|s_n - a| \leq |s_n - s_m| + |s_m - a|.$$

4 Exercises

I. Prove Lemma 2.1

II. Prove Theorem 2.2