Real Analysis

Thomas R. Cameron

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1 Daily Quiz

Define a sequence that is

a. monotone but not Cauchy,

b. Cauchy but not monotone.

2 Key Topics

Today we complete our proof of the fact that all Cauchy sequences are convergent in $\mathbb{R}$. Then, we introduce subsequences. For further reading, see [1, Sections 2.1.3 and 2.4].

Theorem 2.1. *A real sequence is convergent if and only if it is Cauchy.*

Proof. On October 4, we proved that every convergent sequence is Cauchy.

Let $s\colon \mathbb{N} \to \mathbb{R}$ be Cauchy and define its range to be

$$R = \{s_n : \ n \in \mathbb{N}\}.$$

We break the remainder of this proof into two cases: R is finite and R is infinite.

If R is finite, then there exists an $\epsilon > 0$ such that $N(x;\epsilon) \cap R = \{x\}$ for all $x \in R$. Since s is Cauchy, there exists an N such that $n, m > N \Rightarrow |s_n - s_m| < \epsilon$. Fix m as the smallest integer bigger than N. Then, we have

$$n > N \Rightarrow |s_n - s_m| < \epsilon \Rightarrow s_n = s_m.$$

Therefore, $\lim_{n\to\infty} s_n = s_m$.

Suppose R is infinite. Since Cauchy sequences are bounded, the Bolzano-Weirstrass Theorem (see Review 1) implies that R has an accumulation point, which we denote by $a \in \mathbb{R}$. Let $\epsilon > 0$. Then, there is an $N \in \mathbb{R}$ such that

$$n, m > N \Rightarrow |s_n - s_m| < \frac{\epsilon}{2}.$$

Since a is an accumulation point, $N(a;\epsilon/2)$ contains infinitely many points of R. Hence, there is an $m > N$ such that $s_m \in N(a;\epsilon/2)$. Then, for all $n > N$, we have

$$\begin{aligned} |s_n - a| &= |s_n - s_m + s_m - a| \\ &\leq |s_n - s_m| + |s_m - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

2.1 Subsequences

Definition 2.2. Let $s\colon \mathbb{N} \to \mathbb{R}$ and let $n\colon \mathbb{N} \to \mathbb{N}$ be a strictly increasing sequence. Then, the composition $s(n)$ is called a *subsequence* of s, we denote the terms of the subsequence by

$$s_{n_1},\ s_{n_2},\ \ldots,\ s_{n_k},\ \ldots.$$

Example 2.3. Let $s_n = 1/n$. Then, both

$$t_k = 1/(2k) \text{ and } u_k = 1/(2k-1),\ k \in \mathbb{N}.$$

are subsequences of s.

Theorem 2.4. *Let $s\colon \mathbb{N} \to \mathbb{R}$ be convergent with limit L. Then, every subsequence of s also converges to L.*

Proof. Let $n\colon \mathbb{N} \to \mathbb{N}$ be a strictly increasing sequence:

$$n_1 < n_2 < \ldots < n_k < n_{k+1} < \cdots .$$

Then, $n_k \geq k$ for all $k \in \mathbb{N}$.

Let $\epsilon > 0$. Then, there is a $N \in \mathbb{R}$ such that

$$n > N \Rightarrow |s_n - L| < \epsilon.$$

Now,

$$\begin{aligned} k > N &\Rightarrow n_k \geq k > N \\ &\Rightarrow |s_{n_k} - N| < \epsilon. \end{aligned}$$

□

Theorem 2.5. *Every bounded sequence has a convergent subsequence.*

Proof. Let $s\colon \mathbb{N} \to \mathbb{R}$ be a bounded sequence and let

$$R = \{s_n\colon\ n \in \mathbb{N}\}$$

denote the range of s.

If R is finite, then there is an $x \in R$ such that $s_n = x$ for infinitely many n. That is, there exists indices

$$n_1 < n_2 < \cdots < n_k < \cdots$$

such that $s_{n_k} = x$ for all $k \in \mathbb{N}$.

If R is infinite, then the Bolzano-Weierstrass theorem implies that R has an accumulation point, which we denote by $a \in \mathbb{R}$. For each $k \in \mathbb{N}$, there are infinitely many sequence values in the neighborhood

$$A_k = (a - 1/k, a + 1/k)\,.$$

Therefore, we can pick $s_{n_1} \in A_1$, and for $k \geq 2$ choose $s_{n_k} \in A_k$ with $n_k > n_{k-1}$. □

3 Exercises

I. Prove Theorem 2.4.

II. Prove Theorem 2.5.

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.