Real Analysis

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1 Daily Quiz

2 Key Topics

Today we introduce the the Upper and Lower Darboux Sums For further reading, see [1, Section 5.1].

2.1 Upper and Lower Darboux Sums

Definition 2.1. A *partition* P of $[a, b]$ is a finite set $\{x_0, x_1, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

If Q is a partition of $[a, b]$ such that $P \subseteq Q$, we say that Q is a *refinement* of P.

Definition 2.2. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Then, the *upper Darboux sum* of f with respect to P is

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i,$$

where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $\Delta x_i = x_i - x_{i-1}$. Similarly, the *lower Darboux sum* of f with respect to P is

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i,$$

where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. Note that the existence of the infimum and supremum of the set $\{f(x) : x \in [x_{i-1}, x_i]\}$ is guaranteed by the completeness axiom since f is bounded.

Theorem 2.3. *Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. If P and Q are partitions of P, and Q is a refinement of P, then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and let $P^* = P \cup \{x^*\}$, where $x^* \in [a, b] \setminus P$. Then, there exists a $k \in \{1, \ldots, n\}$ such that $x_{k-1} < x^* < x_k$. Now, define

$$t_1 = \inf\{f(x) : x \in [x_{k-1}, x^*]\},$$
$$t_2 = \inf\{f(x) : x \in [x^*, x_k]\}.$$

Then, $t_1 \geq m_k$ and $t_2 \geq m_k$. Therefore,

$$\begin{aligned} L(f, P^*) - L(f, P) &= [t_1(x^* - x_{k-1}) + t_2(x_k - x^*)] - [m_k(x_k - x_{k-1})] \\ &= (t_1 - m_k)(x^* - x_{k-1}) + (t_2 - m_k)(x_k - x^*) \geq 0. \end{aligned}$$

So, $L(f, P^*) \geq L(f, P)$. If Q contains r more points than P, then we apply the above argument r times. Hence,

$$L(f, P) \leq L(f, Q).$$

A similar argument shows that $U(f, Q) \leq U(f, P)$. $\square$

Corollary 2.4. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded. If P and Q are partitions of $[a,b]$, then $L(f,P) \leq U(f,Q)$.*

Proof. Note that $P \cup Q$ is a refinement of both P and Q. Therefore, the previous theorem implies that

$$L(f,P) \leq L(f,P \cup Q) \leq U(f,P \cup Q) \leq U(f,Q).$$

□

2.2 Examples

Example 2.5. Define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Let P be any partition of $[0,1]$. Then,

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} 1 \Delta x_i = (1-0) = 1$$

and

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 \Delta x_i = 0.$$

Example 2.6. Define $f\colon [a,b] \to \mathbb{R}$ by $f(x) = c$, for all $x \in [a,b]$. Let P be any partition of $[a,b]$. Then,

$$\sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} c \Delta x_i = c(b-a)$$

and

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} c \Delta x_i = c(b-a).$$

3 Exercises

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.