Real Analysis

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1 Daily Quiz

2 Key Topics

Today we introduce the Riemann Integral. For further reading, see [1, Section 5.1].

Last time, we introduced the upper and lower Darboux sums:

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i$$

and

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i,$$

where $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$, $M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i]\}$, $m_i = \inf\{f(x)\colon x \in [x_{i-1}, x_i]\}$, and $\Delta x_i = x_i - x_{i-1}$. In addition, we proved the following results.

Theorem 2.1. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded. If P and Q are partitions of P, and Q is a refinement of P, then*

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

Corollary 2.2. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded. If P and Q are partitions of $[a,b]$, then $L(f,P) \leq U(f,Q)$.*

2.1 Riemann Integral

Definition 2.3. Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded. Then, the *upper Darboux integral* of f is

$$\overline{\int_a^b} f(x)dx = U(f) = \inf\{U(f,P)\colon P \text{ is a partition of } [a,b]\}$$

Similarly, the *lower Darboux integral* of f is

$$\underline{\int_a^b} f(x)dx = L(f) = \sup\{L(f,P)\colon P \text{ is a partition of } [a,b]\}$$

Definition 2.4. Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded. If $U(f) = L(f)$, then we say that f is *Riemann integrable*. In this case, we denote the *Riemann integral* of f by

$$\int_a^b f(x)dx = U(f) = L(f).$$

Lemma 2.5. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded. Then, $L(f) \leq U(f)$.*

Proof. Let $A = \{L(f,P) \colon P \text{ is a partition of } [a,b]\}$ and $B = \{U(f,P) \colon P \text{ is a partition of } [a,b]\}$. Then, by Corollary 2.2, $U(f,Q)$ is an upper bound on A for any partition Q of $[a,b]$. Therefore,

$$\sup A \leq U(f,Q)$$

for any partition Q of $[a,b]$. Furthermore, $\sup A$ is a lower bound on B, so it follows that

$$L(f) = \sup A \leq \inf B = U(f).$$

□

Theorem 2.6. *Let $f \colon [a,b] \to \mathbb{R}$ be bounded. Then, f is Riemann integrable if and only if for all $\epsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.*

Proof. Suppose f is Riemann integrable. Then, $L(f) = U(f)$. Let $\epsilon > 0$. Then, there exists a partition P_1 of $[a,b]$ such that

$$L(f,P_1) > L(f) - \frac{\epsilon}{2};$$

otherwise, $L(f)$ is not a least upper bound for the set $\{L(f,P) \colon P \text{ is a partition of } [a,b]\}$. Similarly, there exists a partition P_2 of $[a,b]$ such that

$$U(f,P_2) < U(f) + \frac{\epsilon}{2}.$$

Let $P = P_1 \cup P_2$, then Theorem 2.1 implies that

$$L(f) - \frac{\epsilon}{2} < L(f,P_1) \leq L(f,P) \leq U(f,P) \leq U(f,P_2) < U(f) + \frac{\epsilon}{2}$$

Therefore,

$$\begin{aligned} U(f,P) - L(f,P) &< \left(U(f) + \frac{\epsilon}{2}\right) - \left(L(f) - \frac{\epsilon}{2}\right) \\ &= (U(f) - L(f)) + \epsilon = \epsilon. \end{aligned}$$

Conversely, suppose that for all $\epsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) < L(f,P) + \epsilon$. Then,

$$U(f) \leq U(f,P) < L(f,P) + \epsilon \leq L(f) + \epsilon.$$

Since the above inequality holds for all $\epsilon > 0$, it follows that $U(f) \leq L(f)$. Combined with Lemma 2.5, we have $U(f) = L(f)$ so f is Riemann integrable. □

2.2 Examples

Example 2.7. Define $f \colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Let P be any partition of $[0,1]$. Then,

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} 1 \Delta x_i = (1-0) = 1$$

and

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 \Delta x_i = 0.$$

Therefore, $U(f,P) - L(f,P) = 1$ for all partitions P of $[0,1]$; hence, f is not Riemann integrable.

Example 2.8. Define $f\colon [a,b] \to \mathbb{R}$ by $f(x) = c$, for all $x \in [a,b]$. Let P be any partition of $[a,b]$. Then,

$$\sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} c\Delta x_i = c(b-a)$$

and

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} c\Delta x_i = c(b-a).$$

Therefore, $U(f,P) - L(f,P) = 0$ for all partitions P of $[a,b]$; hence, f is Riemann integrable.

3 Exercises

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.