Real Analysis

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November 15, 2023

1 Daily Quiz

2 Key Topics

Today we discuss the properties of the Riemann Integral. For further reading, see [1, Section 5.2].

Last time, we introduced the upper and lower Darboux integrals:

$$\overline{\int_a^b} f(x)dx = U(f) = \inf\{U(f,P)\colon P \text{ is a partition of } [a,b]\}$$

and

$$\underline{\int_a^b} f(x)dx = L(f) = \sup\{L(f,P)\colon P \text{ is a partition of } [a,b]\},$$

where $U(f,P)$ and $L(f,P)$ are the upper and lower Darboux sums. We say that f is Riemann integrable if $U(f) = L(f)$. In addition, we proved the following result.

Theorem 2.1. *Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Then, f is Riemann integrable if and only if for all $\epsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.*

2.1 Linearity

In this section, we prove that the Riemann integral is a linear transformation over the space of integrable functions.

Theorem 2.2. *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be Riemann integrable functions. Then,*

a. for all $k \in \mathbb{R}$, kf is Riemann integrable and $\int_a^b kf(x)dx = k\int_a^b f(x)dx$,

b. $f+g$ is Riemann integrable and $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

Proof.

a. Let $k \geq 0$ and let P be any partition of $[a,b]$. Then, $U(kf,P) = kU(f,P)$ and $L(kf,P) = kL(f,P)$. Therefore,

$$\begin{aligned} U(kf) &= \inf\{U(kf,P)\colon P \text{ is a partition of } [a,b]\} \\ &= \inf\{kU(f,P)\colon P \text{ is a partition of } [a,b]\} \\ &= k\inf\{U(f,P)\colon P \text{ is a partition of } [a,b]\} \\ &= kU(f). \end{aligned}$$

Similarly, $L(kf) = kL(f)$. Since f is Riemann integrable, it follows that

$$L(kf) = kL(f) = kU(f) = U(kf).$$

Hence, kf is Riemann integrable and $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

For $k < 0$ it is sufficient to show that if f is Riemann integrable, then $-f$ is Riemann integrable and $\int_a^b (-f) = -\int_a^b f$. To this end, note that for any partition P of $[a, b]$, $U(-f, P) = -L(f, P)$ and $L(-f, P) = -U(f, P)$. Therefore, $U(-f) = -L(f)$ and $L(-f) = -L(f)$.

b. Let P be any partition of $[a, b]$. Then, $U(f+g, P) \leq U(f, P)+U(g, P)$ and $L(f+g, P) \geq L(f, P)+L(g, P)$. Let $\epsilon > 0$. By Theorem 2.1, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(f, P_1) < L(f, P_1) + \frac{\epsilon}{2} \text{ and } U(g, P_2) < L(g, P_2) + \frac{\epsilon}{2}.$$

Since $P = P_1 \cup P_2$ is a refinement of both P_1 and P_2, it follows that

$$U(f, P) < L(f, P) + \frac{\epsilon}{2} \text{ and } U(g, P) < L(g, P) + \frac{\epsilon}{2}$$

Therefore, we have

$$U(f + g, P) \leq U(f, P) + U(g, P) < L(f, P) + L(g, P) + \epsilon \leq L(f + g, P) + \epsilon.$$

Hence, Theorem 2.1 implies that $f + g$ is Riemann integrable. Furthermore,

$$\int_a^b (f + g)(x)dx < \int_a^b f(x)dx + \int_a^b g(x)dx + \epsilon$$

and

$$\int_a^b (f + g)(x)dx > \int_a^b f(x)dx + \int_a^b g(x)dx - \epsilon,$$

for all $\epsilon > 0$. So, it follows that

$$\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

□

2.2 Additivity

In this section, we prove that the Riemann integral is additive with respect to its bounds.

Theorem 2.3. *Suppose that $f\colon [a, c] \to \mathbb{R}$ and $f\colon [c, b] \to \mathbb{R}$ is Riemann integrable. Then, $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable and*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Proof. Let $\epsilon > 0$. By Theorem 2.1, there exists partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2} \text{ and } U(f, P_2) - L(f, P_2) < \frac{\epsilon}{2}.$$

Then, $P = P_1 \cup P_2$ is a partition of $[a, b]$ and

$$\begin{aligned} U(f, P) - L(f, P) &= [U(f, P_1) + U(f, P_2)] - [L(f, P_1) + L(f, P_2)] \\ &= [U(f, P_1) - L(f, P_1)] + [U(f, P_2) - L(f, P_2)] \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, Theorem 2.1 implies that $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable. Furthermore,

$$\begin{aligned} \int_a^b f(x)dx &\leq U(f, P) = U(f, P_1) + U(f, P_2) \\ &< L(f, P_1) + L(f, P_2) + \epsilon \\ &\leq \int_a^c f(x)dx + \int_c^b f(x)dx + \epsilon \end{aligned}$$

and

$$\begin{aligned}\int_a^b f(x)dx &\geq L(f,P) = L(f,P_1) + L(f,P_2) \\ &> U(f,P_1) + U(f,P_2) - \epsilon \\ &\geq \int_a^c f(x)dx + \int_c^b f(x)dx - \epsilon,\end{aligned}$$

for all $\epsilon > 0$. So, it follows that

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

□

3 Exercises

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.