Real Analysis

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1 Daily Quiz

2 Key Topics

Today we continue our discussion of the properties of the Riemann integral. For further reading, see [1, Section 5.2].

Recall the Darboux integrals

$$\overline{\int_a^b} f(x)dx = U(f) = \inf\{U(f,P) \colon P \text{ is a partition of } [a,b]\}$$

and

$$\underline{\int_a^b} f(x)dx = L(f) = \sup\{L(f,P) \colon P \text{ is a partition of } [a,b]\},$$

where $U(f,P)$ and $L(f,P)$ are the upper and lower Darboux sums. We say that f is Riemann integrable if $U(f) = L(f)$. In addition, we have the following result.

Theorem 2.1. *Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Then, f is Riemann integrable if and only if for all $\epsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.*

2.1 Monotone and Continuous Functions

In this section, we prove that monotone functions and continuous functions are Riemann integrable.

Theorem 2.2. *Let $f\colon [a,b] \to \mathbb{R}$ be a monotone function. Then, f is Riemann integrable.*

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Note that

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} \left[f(x_i) - f(x_{i-1})\right] \Delta x_i.$$

Let $\epsilon > 0$, then there exists a $k > 0$ such that

$$k\left[f(b) - f(a)\right] < \epsilon.$$

Let P be any partition of $[a,b]$ where $\Delta x_i \leq k$, for $i = 1, \ldots, n$. Then,

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} \left[f(x_i) - f(x_{i-1})\right] \Delta x_i \\ &\leq k \sum_{i=1}^{n} \left[f(x_i) - f(x_{i-1})\right] \\ &= k\left[f(b) - f(a)\right] < \epsilon. \end{aligned}$$

□

Theorem 2.3. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Then, f is Riemann integrable.*

Proof. Let $\epsilon > 0$. Since f is continuous on the compact set $[a,b]$, it follows that f is uniformly continuous. Hence, there is a $\delta > 0$ such that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

Let P be any partition of $[a,b]$ such that $\Delta x_i < \delta$, for $i = 1, \dots, n$. Then,

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} [M_i - m_i]\, \Delta x_i \\ &< \frac{\epsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \epsilon. \end{aligned}$$

□

3 Exercises

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.