Real Analysis

Thomas R. Cameron

November 27, 2023

1 Daily Quiz

2 Key Topics

Today we review properties of the Riemann integral and introduce the fundamental theorem of calculus. For further reading, see [1, Sections 5.2–5.3].

Recall the Darboux integrals

$$\overline{\int_a^b} f(x)dx = U(f) = \inf\{U(f,P)\colon P \text{ is a partition of } [a,b]\}$$

and

$$\underline{\int_a^b} f(x)dx = L(f) = \sup\{L(f,P)\colon P \text{ is a partition of } [a,b]\},$$

where $U(f,P)$ and $L(f,P)$ are the upper and lower Darboux sums. We say that f is Riemann integrable if $U(f) = L(f)$. In addition, we have the following results.

Theorem 2.1. *Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Then, f is Riemann integrable if and only if for all $\epsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.*

Theorem 2.2. *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be Riemann integrable. Then,*

a. $\int_a^b kf = k\int_a^b f,\ \forall k \in \mathbb{R}$

b. $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

Theorem 2.3. *Let $f\colon [a,b] \to \mathbb{R}$ be monotone. Then, f is Riemann integrable.*

Theorem 2.4. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then, f is Riemann integrable.*

2.1 Fundamental Theorem of Calculus: Part I

Theorem 2.5. *Let $f\colon [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$. If f' is Riemann integrable on $[a,b]$, then*

$$\int_a^b f' = f(b) - f(a).$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. By the Mean Value Theorem, for each $i = 1, \ldots, n$, there exists a $c_i \in [x_{i-1}, x_i]$ such that

$$f'(c_i)\Delta x_i = f(x_i) - f(x_{i-1}).$$

Let m_i and M_i denote the infimum and supremum, respectively, of f' on $[x_{i-1}, x_i]$ Then, we have

$$m_i\Delta x_i \leq f'(c_i)\Delta x_i = f(x_i) - f(x_{i-1}) \leq M_i\Delta x_i$$

Since $f(b) - f(a) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))$, it follows that

$$L(f', P) \leq f(b) - f(a) \leq U(f', P).$$

Since this bound holds for any partition P, we have

$$L(f') \leq f(b) - f(a) \leq U(f')$$

Therefore, since f is Riemann integrable,

$$U(f') = L(f') = f(b) - f(a).$$

$\square$

Example 2.6. Let $f(x) = x^3$. Then,

$$\int_a^b 3x^2 dx = b^3 - a^3.$$

3 Exercises

Suppose that $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ are continuously differentiable on $[a,b]$. The following exercise will walk you through the proof of integration by parts.

a. Use the product rule to show that

$$f(x)\frac{d}{dx}g(x) = \frac{d}{dx}(f(x)g(x)) - g(x)\frac{d}{dx}f(x).$$

b. Use Theorem 2.4 to explain why $f(x)\frac{d}{dx}g(x)$ is Riemann integrable on $[a,b]$.

c. Use Theorem 2.2 and Theorem 2.5 to show that

$$\begin{aligned}\int_a^b f(x)\frac{d}{dx}g(x)dx &= \int_a^b \left(\frac{d}{dx}(f(x)g(x)) - g(x)\frac{d}{dx}f(x)\right) dx \\ &= f(x)g(x)\Big|_a^b - \int_a^b g(x)\frac{d}{dx}f(x)dx.\end{aligned}$$

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.