Real Analysis

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1 Daily Quiz

Let $f\colon [a,b] \to \mathbb{R}$ be monotone increasing. Prove that f is Riemann integrable.

2 Key Topics

Today we review the fundamental theorem of calculus, part I, and introduce part II of the theorem. For further reading, see [1, Section 5.3].

Recall part I of the fundamental theorem of calculus.

Theorem 2.1. *Let $f\colon [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$. If f' is Riemann integrable on $[a,b]$, then*

$$\int_a^b f' = f(b) - f(a).$$

2.1 Fundamental Theorem of Calculus: Part II

Theorem 2.2. *Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable. For each $x \in [a,b]$, let*

$$F(x) = \int_a^x f(t)dt.$$

Then, F is uniformly continuous on $[a,b]$. Furthermore, if f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.

Proof. Since f is bounded, there exists an $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. Let $\epsilon > 0$ and $\delta = \epsilon/M$. Let $x, y \in [a,b]$ such that $|x - y| < \delta$. If $x > y$, we have

$$\begin{aligned} |F(x) - F(y)| &= \left| \int_a^x f - \int_a^y f \right| \\ &= \left| \int_y^x f \right| \\ &\leq \int_y^x |f| \\ &\leq \int_y^x M = M(x-y) < \epsilon. \end{aligned}$$

Similarly, if $x < y$, we get $|F(y) - F(x)| < \epsilon$. Hence, F is uniformly continuous on $[a,b]$.

Now, suppose that f is continuous at $c \in [a,b]$. Let $\epsilon > 0$ and let $\delta > 0$ such that for all $x \in [a,b]$, $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$. Let $x \in [a,b]$ such that $0 < |x-c| < \delta$. If $x > c$, we have

$$\begin{aligned}
\left|\frac{F(x)-F(c)}{x-c} - f(c)\right| &= \left|\frac{1}{x-c}\left(\int_a^x f(t)dt - \int_a^c f(t)dt\right) - f(c)\right| \\
&= \left|\frac{1}{x-c}\int_c^x f(t)dt - \frac{1}{x-c}\int_c^x f(c)dt\right| \\
&= \left|\frac{1}{x-c}\int_c^x \left(f(t)-f(c)\right)dt\right| \\
&\leq \frac{1}{|x-c|}\int_c^x |f(t)-f(c)|\,dt < \epsilon.
\end{aligned}$$

Similarly, if $x < c$, we get

$$\left|\frac{F(x)-F(c)}{x-c} - f(c)\right| < \epsilon.$$

Therefore,

$$F'(c) = \lim_{x \to c} \frac{F(x)-F(c)}{x-c} = f(c).$$

□

Example 2.3. Let $f(x) = \sqrt{5+x^3}$ and define

$$F(x) = \int_0^x \sqrt{5+t^3}dt$$

for all $x \in [0,4]$. Then, F is differentiable on $[0,4]$ and $F'(x) = \sqrt{5+x^3}$.

Corollary 2.4. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous and $g\colon [c,d] \to \mathbb{R}$ be differentiable, where $g([c,d]) \subseteq [a,b]$. For each $x \in [c,d]$, let*

$$F(x) = \int_a^{g(x)} f(t)dt.$$

Then, F is differentiable on $[c,d]$ and $F'(x) = f(g(x))g'(x)$.

Example 2.5. Let $f(x) = \sqrt{5+x^3}$ and define

$$F(x) = \int_0^{x^2} \sqrt{5+t^3}dt$$

for all $x \in [0,2]$ Then, F is differentiable on $[0,2]$ and $F'(x) = \sqrt{5+x^6}(2x)$.

3 Exercises

Prove Corollary 2.4.

References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.