Real Analysis

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1 Daily Quiz

2 Key Topics

Consider the heat equation

$$\begin{aligned} u_{xx} &= u_t, \quad -1 < x < 1, \ t > 0 \\ u(-1,t) &= u(1,t) = 0, \ t > 0 \\ u(x,0) &= f(x), \quad -1 \leq x \leq 1 \end{aligned}$$

Suppose the initial temperature distribution can be written as

$$f(x) = a_1 \cos\left(\frac{\pi x}{2}\right) + a_2 \cos\left(\frac{3\pi x}{2}\right) + \cdots + a_n \cos\left(\frac{(2n-1)\pi x}{2}\right). \tag{1}$$

Then, the solution to the heat equation is given by

$$u(x,t) = a_1 e^{-\pi^2 t/4} \cos\left(\frac{\pi x}{2}\right) + a_2 e^{-9\pi^2 t/4} \cos\left(\frac{3\pi x}{2}\right) + \cdots + a_n e^{-(2n-1)^2\pi^2 t/4} \cos\left(\frac{(2n-1)\pi x}{2}\right). \tag{2}$$

A sample solution is shown in Figure 1.

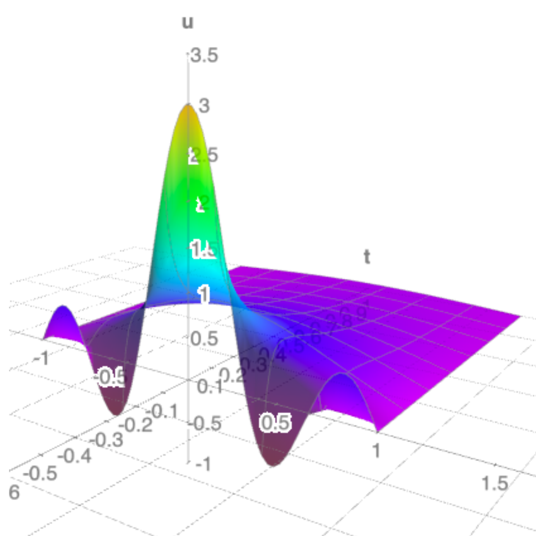

Figure 1: Sample Heat Equation Solution

In 1807, Joseph Fourier demonstrated that, for $-1 < x < 1$, we have

$$1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos\left(\frac{(2n-1)\pi x}{2}\right). \tag{3}$$

Hence, the heat equation could be solved with initial temperature distributions that are constant. At the time, some of the greatest mathematicians had dismissed the use of infinite series of trigonometric functions due to the logical danger they presented. For example, see Exercise III.

The need for "more than calculus" was born out of the problems presented by Joseph Fourier and other mathematicians whose intuition could not be rigorously explained. Over the next 100 years, mathematicians such as Joseph Fourier, Niels Henrik Abel, Augustin-Louis Cauchy, and Gustav Lejeune Dirichlet would lay the foundation for what is now known as real analysis.

3 Exercises

I. Verify that the solution given in (2) satisfies the heat equation.

II. What goes wrong if we change the initial temperature distribution to $f(x) = 1$?

III. Show that term by term differentiation of the infinite series in (3) gives

$$-2\sum_{n=1}^{\infty}(-1)^{n-1} \sin\left(\frac{(2n-1)\pi x}{2}\right),$$

which only converges to zero when x is an even integer.

References