Real Analysis

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1 Daily Quiz

2 Key Topics

Today we introduce basic logic which forms the foundation for the language of mathematics. For additional information, see [1, Chapter 2]. The language of mathematics is built upon *statements*, which are sentences that can be classified as true or false. *Logical connectives*, such as *not*, *and*, *or*, *if*, *then*, *if and only if* play an important role as they are the building blocks of compound statements.

Example 2.1. The following statements all have varying meaning.

i. It is windy and the waves are high.

ii. It is windy or the waves are high.

iii. If it is windy, then the waves are high.

iv. It is windy if and only if the waves are high.

One way to study the different meaning between varying statements is to study their truth table. To that end, let $p =$ "it is windy" and $q =$ "the waves are high". Then, each statement above is represented in the following truth table.

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Note that the statement $p \Leftrightarrow q$ is called an *equivalence* since the compound statement is true only when p and q take on the same truth value.

Consider the sentence

$$x^2 - 5x + 6 = 0,$$

which requires the variable x to be quantified in order to become a statement. In particular, there are some x values for which the sentence is true and other x values for which the sentence is false. Therefore, the following is a true statement:

$$\text{there exists an } x \text{ such that } x^2 - 5x + 6 = 0$$

and the following is a false statement:

$$\text{for all } x, \text{ we have } x^2 - 5x + 6 = 0.$$

The existential quantifier (there exists) and the universal quantifier (for all) are so common that they get their own notation: $\exists$ and $\forall$, respectively. In addition, we use $\ni$ as a shorthand for the phrase "such that", Therefore, the above statements can be re-written as follows

$$\exists x \ \ni \ x^2 - 5x + 6 = 0$$

and

$$\forall x, \ x^2 - 5x + 6 = 0,$$

respectively.

The negation of quantified statements requires careful consideration. For example, let $f(x)$ be a real-valued function and consider the statement

$$\forall x, \ f(x) > 5.$$

For this statement to be false, there must exist at least one x such that $f(x) \leq 5$. Hence, the negation can be written as follows

$$\neg[\forall x, \ f(x) > 5] \Leftrightarrow \exists x \ \ni \ f(x) \leq 5,$$

where $\neg$ is the symbol used for the *negation* of a statement.

3 Exercises

I. Write a truth table for the statements $\neg(p \wedge q)$ and $\neg p \vee \neg q$.

II. Write a truth table for the statements $p \Rightarrow q$ and $\neg p \vee q$.

III. Rewrite each statement using $\exists$, $\forall$, and $\ni$ as appropriate

   a. There exists a positive number x such that $x^2 = 5$.
   b. For every positive number M there is a positive number N such that $N < 1/M$.
   c. If $n \geq N$, then $|f_n(x) - f(x)| \leq 3$ for all $x \in A$.

IV. Write the negation of each statement in II.

References

[1] R. Hammack, *Book of Proof*, Creative Commons Attribution-NonCommercial-NoDerivative, 3rd ed., 2018.