Real Analysis

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1 Daily Quiz

Use a truth table to determine if the implications $p \Rightarrow q$ and $\neg q \Rightarrow \neg q$ are equivalent.

2 Key Topics

A *theorem* is a mathematical statement that is true and can be verified as true. A *proof* is the written verification that a theorem is true. Today we introduce several proof techniques that will be used throughout the semester. In particular, we will cover the direct proof, contrapositive proof, and the proof by contradiction. For further reading, see [1, Chapters 4–6].

A statement that is true but not as significant is sometimes called a *proposition* A *lemma* is a theorem whose main purpose is to help prove another theorem. A *corollary* is a theorem that is an immediate consequence of another theorem or proposition.

2.1 Direct Proof

Let p and q be two statements. In the implication $p \Rightarrow q$, p is the hypothesis and q is the conclusion. If $p \Rightarrow q$ is true, then a direct proof would verify this by assuming the hypothesis and then constructing a sequence of true statements until the conclusion is reached.

Recall the definition of even and odd integers.

Definition 2.1. An integer n is *even* if $n = 2k$ for some other integer k. An integer n is *odd* if $n = 2k + 1$ for some other integer k.

Also, recall the definition of divides.

Definition 2.2. The integer a *divides* the integer b, written $a|b$, if $b = ac$ for some integer c. In this case, we say that a is a *divisor* of b, b is a *multiple* of a, or that b is *divisible* by a.

Below are several propositions that can be proven directly.

Proposition 2.3. *If n and m are odd, then $n + m$ is even.*

Proposition 2.4. *Let a, b, c, d be integers. If $a|b$ and $c|d$, then $ac|bd$.*

2.2 Contrapositive Proof

Associated with an implication $p \Rightarrow q$ is a related implication $\neg q \Rightarrow \neg p$ called the *contrapositive.* The following truth table shows that these compound statements are equivalent.

Below are several propositions for which a contrapositive proof is easier than a direct proof.

Proposition 2.5. *Let a be an integer. If a^2 is not divisible by 4, then a is odd.*

Proposition 2.6. *If p^2 is odd, then p is odd.*

p	q	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

2.3 Proof by Contradiction

To prove a statement p is true we can also use a proof by contradiction, which is based on the following equivalence

$$(\neg p \Rightarrow c) \Leftrightarrow p,$$

where c is a *contradiction*, i.e., a statement that is always false. The above equivalence is illustrated in the following truth table.

p	c	$\neg p \Rightarrow c$
T	F	T
F	F	T

Recall the equivalence

$$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)\,.$$

Therefore, we have

$$(p \Rightarrow q) \Leftrightarrow [(p \wedge \neg q) \Rightarrow c]\,,$$

which allows us to prove implications by contradiction.

Recall the definition of rational and irrational numbers.

Definition 2.7. A real number x is *rational* if there exists integers m and n such that $x = m/n$. If x is not rational, then we say it is *irrational*.

Below are several propositions for which proof by contradiction is the easiest.

Proposition 2.8. *$\sqrt{3}$ is irrational.*

Proposition 2.9. *If x is rational and y is irrational, then $x + y$ is irrational.*

3 Exercises

I. Prove Propositions 2.3–2.4.

II. Prove Propositions 2.5–2.6.

III. Prove Propositions 2.8–2.9.

References

[1] R. Hammack, *Book of Proof*, Creative Commons Attribution-NonCommercial-NoDerivative, 3rd ed., 2018.