Real Analysis

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1 Daily Quiz

Prove the following statement: If $x = 3k + 1$, for some integer k, then x^2 is not divisible by 3.

2 Key Topics

Today we revisit Propositions 2.8–2.9 from 8/25/2023. Then, we introduce sets and basic set operations. For further reading, see [1, Sections 1.1, 1.3, 1.5–1.6].

2.1 Sets

A *set* is a collection of things, which are referred to as *elements* of the set. It is customary to use capital letters to designate a set, the symbol $\in$ to denote those in the set, and $\notin$ for those not in the set. For example, if $A = \{1, 2, 3, 4\}$, then $2 \in A$ and $5 \notin A$.

Definition 2.1. Let A and B be sets. We say that A is a *subset* of B, denoted $A \subseteq B$, if $\forall x \in A$, $x \in B$. If $A \subseteq B$ and $\exists x \in B \ni x \notin A$, then A is a *proper subset* of B, which we denote by $A \subset B$.

Definition 2.2. Let A and B be sets. We say that A is *equal* to B, written $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Definition 2.3. We let $\emptyset$ denote the *empty set*, $\mathbb{N}$ the set of *natural numbers*, $\mathbb{Z}$ the set of *integers*, $\mathbb{Q}$ the set of all *rational numbers*, and $\mathbb{R}$ the set of *real numbers*.

Definition 2.4. We define a *closed interval* by

$$[a, b] = \{x \in \mathbb{R} \colon \ a \leq x \leq b\}$$

a *open interval* by

$$(a, b) = \{x \in \mathbb{R} \colon \ a < x < b\}$$

and a *half-open interval* (or *half-closed interval*) by

$$[a, b) = \{x \in \mathbb{R} \colon \ a \leq x < b\} \text{ or } (a, b] = \{x \in \mathbb{R} \colon \ a < x \leq b\}.$$

Example 2.5. Let $A = \{x \in \mathbb{Z} \colon \ 4|x\}$ and let $B = \{x \in \mathbb{Z} \colon \ 2|x\}$. For every $x \in A$ there exists an integer k such that $x = 4k$. Therefore, $x = 2(2k)$, which implies that $x \in B$. Hence, $A \subseteq B$. Note that $A \neq B$ since $2 \in B$ and $2 \notin A$.

2.2 Basic Set Operations

In this section, we review the basic set operations, which allow us to combine sets to create new sets.

Definition 2.6. Let A and B be sets. The *union* of A and B (denoted $A \cup B$), the *intersection* of A and B (denote $A \cap B$), and the *complement* of B *in* A (denoted $A \setminus B$) are given by

$$\begin{aligned} A \cup B &= \{x\colon\; x \in A \vee x \in B\} \\ A \cap B &= \{x\colon\; x \in A \wedge x \in B\} \\ A \setminus B &= \{x\colon\; x \in A \wedge x \notin B\} \end{aligned}$$

If A is a universal set, then the complement of B in A is denoted $\overline{B}$ and referred to as the *complement* of B.

Definition 2.7. Let A and B be sets. A and B are said to be *disjoint* provided that $A \cap B = \emptyset$.

We often want to perform set operations over an entire set of sets. To this end, let $\mathcal{F}$ denote a set of sets. Then, the union and intersection over the entire set is defined as follows

$$\bigcup_{A \in \mathcal{F}} A = \{x\colon\; \exists A \in \mathcal{F} \;\ni\; x \in A\}$$

and

$$\bigcap_{A \in \mathcal{F}} A = \{x\colon\; \forall A \in \mathcal{F},\; x \in A\}\,.$$

Theorem 2.8. *Let A, B, and C be sets. Then, the following statements are true.*

a. $A \cap \overline{B} = A \setminus B$

b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

c. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

d. $\overline{B \cup C} = \overline{B} \cap \overline{C}$

e. $\overline{B \cap C} = \overline{B} \cup \overline{C}$

3 Exercises

I. Prove Theorem 2.8.

II. Define $\mathcal{F} = \{(-1/x, 1/x)\colon\; x \in \mathbb{R} \wedge x > 0\}$. Find

$$\bigcap_{A \in \mathcal{F}} A.$$

References

[1] R. Hammack, *Book of Proof*, Creative Commons Attribution-NonCommercial-NoDerivative, 3rd ed., 2018.