Real Analysis

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1 Daily Quiz

Let A and B be sets. Prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

2 Key Topics

Today we introduce ordered pairs and Cartesian products which we use to define relations. For further reading, see [1, Chapter 11].

2.1 Cartesian Products

Definition 2.1. The *ordered pair* (a, b) is an ordered set of two elements where

$$(a, b) = (c, d) \Leftrightarrow a = c \ \wedge \ b = d.$$

The following theorem shows that we can define an ordered pair as a set of sets.

Theorem 2.2. *Let*

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

Then, $(a, b) = (c, d)$ *if and only if* $a = c$ *and* $b = d$*.*

Definition 2.3. Let A and B be sets. The *Cartesian product* of A and B, written $A \times B$, is defined by

$$A \times B = \{(a, b)\colon a \in A \ \wedge \ b \in B\}$$

Example 2.4. Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then,

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}.$$

Example 2.5. Let $A = [1, 4)$ and $B = (2, 4]$. Then, $A \times B = \{(x, y)\colon \ 1 \le x < 4 \ \wedge \ 2 < y \le 4\}$.

2.2 Relations

Definition 2.6. Let A and B be sets. A *relation* between A and B is any subset $R \subseteq A \times B$. We say that $a \in A$ and $b \in B$ are *related* by R if $(a, b) \in R$, which we often denote by aRb.

Definition 2.7. Let S be a set. A relation $R \subseteq S \times S$ on S is an *equivalence relation* if $\forall x, y, z \in S$ the following properties hold:

a. xRx (reflexive property),

b. $xRy \Rightarrow yRx$ (symmetric property),

c. $(xRy \ \wedge \ yRz) \Rightarrow xRz$ (transitive property).

Definition 2.8. Let $n \in \mathbb{N}$. Define the *congruence modulo* n relation on $\mathbb{Z}$ as follows

$$a \equiv b \mod n$$

if $n|(a-b)$.

Theorem 2.9. *Let $n \in \mathbb{N}$. Then, the congruence modulo n relation is an equivalence relation.*

Given an equivalence relation R on a set S, it is natural to group together all the elements that are related to a particular element. More precisely, we define the *equivalence class* (with respect to R) of $x \in S$ as

$$E_x = \{y \in S\colon \ yRx\}.$$

Example 2.10. Let R denote the congruence modulo 2 relation on $\mathbb{Z}$. Then,

$$\begin{aligned} E_0 &= \{y \in \mathbb{Z}\colon \ yR0\} \\ &= \{y \in \mathbb{Z}\colon \ 2|(y-0)\} \\ &= \{y \in \mathbb{Z}\colon \ \exists k \in \mathbb{Z} \ni \ y = 2k\} \end{aligned}$$

and similarly

$$E_1 = \{y \in \mathbb{Z}\colon \ \exists k \in \mathbb{Z} \ni \ y = 2k+1\}.$$

Hence, the equivalence classes E_0 and E_1 are the set of even and odd integers, respectively.

Definition 2.11. A *partition* of a set S is a set $\mathcal{P}$ of non-empty subsets of S such that

a. $\forall x \in S, \ \exists A \in \mathcal{P} \ni \ x \in A$

b. $\forall A, B \in \mathcal{P}, \ A \neq B \Rightarrow A \cap B = \emptyset$.

Any member of the set $\mathcal{P}$ is called a *piece* of the partition.

Theorem 2.12. *Let R be an equivalence relation on a set S. Then, $\{E_x\colon \ x \in S\}$ is a partition of S. Conversely, if $\mathcal{P}$ is a partition of S, let R be defined by xRy if and only if x and y are in the same piece of the partition. Then, R is an equivalence relation and the corresponding partition into equivalence classes is the same as $\mathcal{P}$.*

3 Exercises

I. Prove Theorem 2.9.

II. Prove Theorem 2.12.

III. Use the Euclid division lemma and Theorem 2.12 to prove that

$$\mathcal{P} = \{E_0, E_1\}$$

is a partition of $\mathbb{Z}$, where E_0 and E_1 are the equivalence classes from Example 2.10.

References

[1] R. Hammack, *Book of Proof*, Creative Commons Attribution-NonCommercial-NoDerivative, 3rd ed., 2018.