Real Analysis

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1 Daily Quiz

State the well-ordering property of $\mathbb{N}$ and the principle of mathematical induction.

2 Key Topics

Today we finish our discussion of induction and introduce the notion of an ordered field. For further reading, see [1, Section 1.1].

2.1 Mathematical Induction

On September 8 2023, we proved that the well-ordering principle of $\mathbb{N}$ implies the principle of mathematical induction. Now, we prove that the principle of mathematical induction implies the well-ordering principle of $\mathbb{N}$.

Theorem 2.1. *If the principle of mathematical induction holds, then the well-ordering principle of* $\mathbb{N}$ *holds.*

Proof. For the sake of contradiction, suppose that there exists a non-empty subsets $S \subseteq \mathbb{N}$ that has no minimal element. Let $P(n)$ be the statement $n \notin S$ for all $n \in \mathbb{N}$. Clearly $P(1)$ is true since if $1 \in S$ then 1 would be the minimal element of S. Since the principle of mathematical induction holds, it follows that

$$P(1),\ P(2),\ P(3),\ \ldots$$

is true. Therefore, $S = \emptyset$, which is a contradiction. □

2.2 Ordered Fields

We begin by assuming the existence of a set $\mathbb{R}$, called the set of real numbers, and two operations $+$ and $\cdot$, called addition and multiplication, respectively, such that the following axioms hold.

Axiom 2.2 (Field $\mathbb{R}$)**.**

a. $\forall x, y \in \mathbb{R},\ x + y \in \mathbb{R}$

b. $\forall x, y \in \mathbb{R},\ x + y = y + x$

c. $\forall x, y, z \in \mathbb{R},\ x + (y + z) = (x + y) + z$

d. There is a unique $0 \in \mathbb{R}$ such that $x + 0 = x,\ \forall x \in \mathbb{R}$

e. $\forall x \in \mathbb{R},\ \exists - x \in \mathbb{R}\ \ni\ x + (-x) = 0$

f. $\forall x, y \in \mathbb{R},\ x \cdot y \in \mathbb{R}$

g. $\forall x, y, z \in \mathbb{R},\ x \cdot (y \cdot z) = (x \cdot y) \cdot z$

h. There is a unique $1 \in \mathbb{R}$ such that $x \cdot 1 = x,\ \forall x \in \mathbb{R}$

i. $\forall x \in \mathbb{R} \setminus \{0\},\ \exists 1/x \in \mathbb{R} \ \ni\ x \cdot (1/x) = 1$

j. $\forall x, y, z \in \mathbb{R},\ x \cdot (y + z) = x \cdot y + x \cdot z$

Axiom 2.3 (Ordering $\mathbb{R}$)**.**

a. For all $x, y \in \mathbb{R}$, exactly one of the relations $x = y$, $x > y$, $x < y$ holds

b. $\forall x, y, z \in \mathbb{R},\ (x < y) \wedge (y < z) \Rightarrow x < z$

c. $\forall x, y, z \in \mathbb{R},\ x < y \Rightarrow x + z < y + z$

d. $\forall x, y, z \in \mathbb{R},\ (x < y) \wedge (z > 0) \Rightarrow xz < yz$

Assuming Axioms 2.2 and 2.3 give us the following useful properties of the real numbers.

Theorem 2.4. *Let $x, y, z \in \mathbb{R}$. Then*

a. $(x + y = y + z) \Rightarrow x = y$

b. $x \cdot 0 = 0$

c. $(-1) \cdot x = -x$

d. $xy = 0 \Leftrightarrow (x = 0 \vee y = 0)$

e. $x < y \Leftrightarrow -y < -x.$

f. $(x < y \wedge z < 0) \Rightarrow (xz > yz)$

Theorem 2.5. *Let $x, y \in \mathbb{R}$ such that $x \leq y + \epsilon,\ \forall \epsilon > 0$. Then $x \leq y$.*

Definition 2.6. Let $x \in \mathbb{R}$. Then the *absolute value* of x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Theorem 2.7. *Let $x, y, a \in \mathbb{R}$, where $a \geq 0$. Then*

a. $|x| \geq 0$

b. $|x| \leq a \Leftrightarrow -a \leq x \leq a$

c. $|xy| = |x| \cdot |y|$

d. $|x + y| \leq |x| + |y|$

3 Exercises

I. Prove Theorem 2.4

II. Prove Theorem 2.5

III. Prove Theorem 2.7

References

[1] W. Trench, *Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 2nd ed., 2013.