Real Analysis

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1 Daily Quiz

State which properties (reflexive, symmetric, or transitive) each of the following relations have.

a. $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$ on $\{1,2,3\}$

b. $R = |$ on $\mathbb{Z}$.

2 Key Topics

Today we introduce a general notion of functions as a relation between two sets. For further reading, see [1, Chapter 12].

Definition 2.1. Let A and B be sets. A *function* between A and B is an nonempty relation $f \subseteq A \times B$ such that if $(a,b) \in f$ and $(a,b') \in f$ then $b = b'$. The *domain* and *range* of f, denoted $\operatorname{dom}(f)$ and $\operatorname{rng}(F)$, respectively, is defined as follows

$$\begin{aligned} \operatorname{dom}(f) &= \{a \in A\colon\ \exists b \in B \ \ni \ (a,b) \in f\} \\ \operatorname{rng}(f) &= \{b \in B\colon\ \exists a \in A \ \ni \ (a,b) \in f\}. \end{aligned}$$

The set B is referred to as the *codomain* of f. If the domain of f contains all of A, then we say f is a *function from A into B* and w write $f\colon\ A \to B$.

2.1 Properties of Functions

Definition 2.2. A function $f\colon\ A \to B$ is *surjective* if $B = \operatorname{rng}(f)$ and *injective* if

$$\forall a, a' \in A,\ f(a) = f(a') \Rightarrow a = a'.$$

Moreover, f is *bijective* if it is both surjective and injective.

Example 2.3. Define $f(x) = x^2$. Then,

$$f\colon\ \mathbb{R} \to \mathbb{R}$$

is neither surjective nor injective,

$$f\colon\ \mathbb{R} \to [0,\infty)$$

is surjective but not injective,

$$f\colon\ [0,\infty) \to [0,\infty)$$

is surjective and bijective.

Definition 2.4. Let $f\colon\ A \to B$, $C \subseteq A$, and $D \subseteq B$. The *image* of C in B is defined by

$$f(C) = \{f(x)\colon\ x \in C\}$$

and the *pre-image* of D in A is defined by

$$f^{-1}(D) = \{x \in A\colon\ f(x) \in D\}.$$

Theorem 2.5. *Let $f\colon A \to B$, $C \subseteq A$, and $D \subseteq B$.*

a. If f is injective, then $f^{-1}(f(C)) = C$.

b. If f is surjective, then $f(f^{-1}(D)) = D$.

2.2 The Inverse Function

Definition 2.6. Let $f\colon A \to B$ be bijective. The *inverse function* $f^{-1}\colon B \to A$ is defined by

$$f^{-1} = \{(b,a) \in B \times A\colon (a,b) \in f\}. \tag{1}$$

Theorem 2.7. *The inverse function $f^{-1}\colon B \to A$ is bijective.*

Definition 2.8. Let $f\colon A \to B$ and $g\colon B \to C$. The *composition* of f and g is defined by

$$g \circ f = \{(a,c) \in A \times C\colon \exists b \in B \ni (a,b) \in f \wedge (b,c) \in g\}$$

Definition 2.9. Let A be a set. The identity function on A is defined by

$$i_A = \{(a,a)\colon a \in A\}.$$

Theorem 2.10. *Let $f\colon A \to B$ be bijective and let f^{-1} be the inverse function. Then,*

$$f^{-1} \circ f = i_A \text{ and } f \circ f^{-1} = i_B.$$

3 Exercises

I. Prove Theorem 2.5

II. Prove Theorem 2.7

III. Prove Theorem 2.10

References

[1] R. Hammack, *Book of Proof*, Creative Commons Attribution-NonCommercial-NoDerivative, 3rd ed., 2018.