Real Analysis

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1 Daily Quiz

Give an example of an infinite set of numbers that is not a field. Justify your answer. Can you identify a field that is not ordered?

2 Key Topics

On 9/11/2023 we introduced the field axiom and ordered axiom, both of which are satisfied by the real numbers. However, these properties are not enough to define the real numbers since these properties are also satisfied by the rational numbers. Today we discuss the completeness axiom, which is the property that distinguishes $\mathbb{R}$ from $\mathbb{Q}$. For further reading, see [1, Section 1.1].

2.1 Upper/Lower Bounds and Suprema/Infimum

Definition 2.1. Let $S \subseteq \mathbb{R}$. If $m \in \mathbb{R}$ satisfies $m \geq s$ for all $s \in S$, then m is an *upper bound* for S. If $m \in \mathbb{R}$ satisfies $m \leq s$ for all $s \in S$, then m is a *lower bound* for S. The set S is *bounded* if it has an upper and lower bound. If an upper bound is an element of S, then we call it a maximum of S. If a lower bound is an element of S, then we call it a minimum of S.

Example 2.2.

a. Let $S = \{2, 4, 6, 8\}$. Then, S has many upper bounds, e.g., $8,\ 9,\ \pi^2$. Since $8 \in S$, $\max S = 8$. Similarly, S has many lower bounds, e.g., $2, 1, 0$. Since $2 \in S$, $\min S = 2$.

b. Let $S = [1, \infty)$. Then, S is not bounded above. However, S is bounded below and $\min S = 1$.

c. Let $S = (0, 4]$. Then, S is bounded above and $\max S = 4$. While S is bounded below, it has no minimum since its greatest lower bound is not an element of S.

Definition 2.3. Let $S \subseteq \mathbb{R}$. If S is bounded above, then the least upper bound of S is called its supremum, denoted $\sup S$. Specifically, $m = \sup S$ provided that

a. $m \geq s,\ \forall s \in S$,

b. $m' < m \ \Rightarrow\ \exists s' \in S \ \ni\ s' > m'$.

Definition 2.4. Let $S \subseteq \mathbb{R}$. If S is bounded below, then the greatest lower bound of S is called its infimum, denoted $\inf S$. Specifically, $m = \inf S$ provided that

a. $m \leq s,\ \forall s \in S$,

b. $m' > m \ \Rightarrow\ \exists s' \in S \ \ni\ s' < m'$.

It is not clear whether a set that is bounded above has a supremum. Indeed, the set $T = \left\{q \in \mathbb{Q}\colon\ 0 \leq q \leq \sqrt{2}\right\}$ does not have a supremum when considered a subset of $\mathbb{Q}$. However, the set T does have a supremum when considered a subset of $\mathbb{R}$. In fact, every subset of $\mathbb{R}$ that is bounded above has a supremum, which is the crux of the completeness axiom.

Axiom 2.5 (Completeness of $\mathbb{R}$)**.** Let $S \subseteq \mathbb{R}$ be non-empty. If S is bounded above, then $\sup S$ exists in $\mathbb{R}$.

While the completeness axiom only refers to sets that are bounded above, the corresponding property for sets bounded below follows readily.

Proposition 2.6. *Let $S \subseteq \mathbb{R}$ be non-empty. If S is bounded below, then $T = \{-s\colon s \in S\}$ is bounded above. If $m = \sup T$, then $-m = \inf S$.*

Not only are the infimum and supremum guaranteed to exist for sets that are bounded below and above, respectively, these values are also unique.

Proposition 2.7. *Let $S \subseteq \mathbb{R}$. If S is bounded above, then $m = \sup S$ is unique.*

The following two theorems illustrate techniques for working with suuprema. Clearly, there are analogous results that hold for infima.

Theorem 2.8. *Let A and B be non-empty subsets of $\mathbb{R}$ and define*

$$C = \{x + y\colon \ x \in A \ \wedge \ y \in B\}.$$

If A and B have suprema, then C has a suprema and

$$\sup C = \sup A + \sup B.$$

Theorem 2.9. *Suppose that $D \subseteq \mathbb{R}$ is non-empty and that $f\colon \ D \to \mathbb{R}$ and $g\colon \ D \to \mathbb{R}$. If for all $x, y \in D$, $f(x) \leq g(y)$, then $f(D)$ is bounded above and $g(D)$ is bounded below. Furthermore, $\sup f(D) \leq \inf g(D)$.*

3 Exercises

I. Prove Proposition 2.6

II. Prove Proposition 2.7

III. Prove Theorem 2.8

IV. Prove Theorem 2.9

References

[1] W. Trench, *Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 2nd ed., 2013.