Real Analysis

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1 Daily Quiz

For each subset of $\mathbb{R}$, give its supremum and its maximum, if they exist.

a. $\{1, 3\}$

b. $(0, 4)$

c. $\left\{1 - \frac{1}{n} : \ n \in \mathbb{N}\right\}$

2 Key Topics

On 9/13/2023, we introduced the completeness axiom which is the additional property that distinguishes the real numbers from the rational numbers. Today, we continue our investigation of the completeness axiom, with a focus on proving theorems that rely on this property of the real numbers. For further reading, see [1, Section 1.1].

2.1 Archimedean Property

One important consequence of the completeness axiom is the Archimedean property, which states that the natural numbers are not bounded above in the real numbers.

Theorem 2.1 (Archimedean Property of $\mathbb{R}$). *The set $\mathbb{N}$ is unbounded above in $\mathbb{R}$.*

Proof. For the sake of contradiction, suppose that $\mathbb{N}$ is bounded above. Then, the completeness axiom states that $\mathbb{N}$ has a least upper bound, which we denote by $\sup \mathbb{N}$. Therefore, there exists an $n \in \mathbb{N}$ such that $n > \sup \mathbb{N} - 1$. However, since $(n+1) \in \mathbb{N}$, we have the contradiction $(n+1) > \sup \mathbb{N}$. $\square$

Theorem 2.2. *Each of the following is equivalent to the Archimedean Property.*

a. $\forall z \in \mathbb{R},\ \exists n \in \mathbb{N} \ \ni \ n > z.$

b. $\forall x, y \in \mathbb{R}\ (x > 0),\ \exists n \in \mathbb{N} \ \ni \ nx > y.$

c. $\forall x \in \mathbb{R}\ (x > 0),\ \exists n \in \mathbb{N} \ \ni \ 0 < 1/n < x.$

Proof. Assume that Theorem 2.1 holds. If a. is false, then there exists a $z \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $n < z$. However, this contradicts $\mathbb{N}$ being unbounded above in $\mathbb{R}$. If a. is true, then b. follows by setting $z = y/x$. Indeed, $n > y/x$ implies that $xn > y$. If b. is true, then c. follows by setting $y = 1$. Indeed, $nx > 1$ implies that $x > 1/n > 0$. Assume that c. is true. If Theorem 2.1 does not hold, then there is a $x \in \mathbb{R}$ such that $x > 0$ and $x > n$ for all $n \in \mathbb{N}$. However, this contradicts c. since it follows $1/n > 1/x$ for all $n \in \mathbb{N}$. $\square$

We can use the Archimedean property of $\mathbb{R}$ and the well-ordering property of $\mathbb{N}$ to prove the density of the rational numbers in $\mathbb{R}$.

Theorem 2.3 (Density of $\mathbb{Q}$ in $\mathbb{R}$). *If $x, y \in \mathbb{R}$ with $x < y$, then there exists a rational r such that $x < r < y$.*

Proof. Theorem 2.2(a.) implies that there exists a $n \in \mathbb{R}$ such that

$$n > \frac{1}{y-x} \;\Rightarrow\; ny > 1 + nx.$$

If $nx \geq 0$, then the following subset

$$\{m \in \mathbb{N} \colon \; m > nx\}$$

has a minimal element by the well-ordering property. Similarly, if $nx < 0$, then the following subset

$$\{m \in \mathbb{N} \cup \{0\} \colon \; -m > nx\}$$

has a minimal element. In either case, there is a $m \in \mathbb{Z}$ such that

$$m - 1 \leq nx < m \;\Rightarrow\; nx < m < ny$$

Therefore,

$$x < \frac{m}{n} < y.$$

□

One can use the above theorem to prove the density of the irrational numbers in $\mathbb{R}$. Finally, we can use the Archimedean property of $\mathbb{R}$ and the well-ordering property of $\mathbb{N}$ to prove Euclid's division lemma.

Theorem 2.4 (Euclid's Division Lemma)**.** *For all $a, b, \in \mathbb{N}$, there exists unique $q, r \in \mathbb{N}$ such that $r < a$ and $b = qa + r$.*

Proof. Let $S = \{m \in \mathbb{N} \colon \; ma > b\}$. By Theorem 2.2(b.), S is non-empty. Therefore, the well-ordering property implies that S has a minimal element, which we denote m. Now, define $q = m - 1$ and $r = b - qa$. Then, clearly $b = qa + r$. Furthermore, since m is the minimal element of S, it follows that q and r are unique and

$$a = ma - (m-1)a > b - qa = r$$

□

3 Exercises

I. Prove Theorem 2.1

II. Prove Theorem 2.2

III. Prove Theorem 2.3

IV. Prove Theorem 2.4

References

[1] W. Trench, *Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 2nd ed., 2013.