Real Analysis

Thomas R. Cameron

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1 Daily Quiz

Let $a, b \in \mathbb{N}$. Explain why the following set has a minimal element:

$$S = \{m \in \mathbb{N}: \ ma > b\}.$$

2 Key Topics

On 9/13/2023, we introduced the completeness axiom which is the additional property that distinguishes the real numbers from the rational numbers. Today, we continue our investigation of the real numbers by studying closed and open sets and accumulation points. For further reading, see [1, Section 1.3].

2.1 Closed and Open Sets

Definition 2.1. Let $x \in \mathbb{R}$ and let $\epsilon > 0$. The *ϵ-neighborhood* of x is defined by

$$N(x;\epsilon) = \{y \in \mathbb{R}: \ |x-y| < \epsilon\}.$$

The *deleted neighborhood* is

$$N^*(x;\epsilon) = N(x;\epsilon) \setminus \{x\} = \{y \in \mathbb{R}: \ 0 < |x-y| < \epsilon\}.$$

Definition 2.2. Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is a *interior point* of S if

$$\exists \epsilon > 0 \ \ni \ N(x;\epsilon) \subseteq S.$$

We say that x is a *boundary point* of S if

$$\forall \epsilon > 0, \ N(x;\epsilon) \cap S \neq \emptyset \ \wedge \ N(x;\epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset.$$

The set of all interior points of S is denoted $\text{int}(S)$ and the set of all boundary points of S is denoted $\text{bd}(S)$.

Definition 2.3. Let $S \subseteq \mathbb{R}$. If $\text{bd}(S) \subseteq \mathbb{S}$, then S is said to be *closed.* If $\text{bd}(S) \subseteq \mathbb{R} \setminus S$, then S is said to be *open.*

Note that a set S is open if and only if every point in S is an interior point of S. Also, a set S is closed if and only if its complement $\mathbb{R} \setminus S$ is open. Finally, the concept of being open or closed is not mutually exclusive, e.g., the empty set $\emptyset$ and its complement $\mathbb{R}$ are both open and closed. Moreover, a subset may be neither open nor closed, i.e.g, the interval $S = (0, 4]$ is not closed since $\text{bd}(S) = \{0, 4\}$ and $0 \notin S$, and S is not open since $4 \in S$.

Theorem 2.4.

a. The union of any collection of open sets is an open set.

b. The intersection of any finite collection of open sets is an open set.

Proof.

a. Let $\mathcal{F}$ be a set of open sets and define

$$S = \bigcup_{A \in \mathcal{F}} A.$$

Let $x \in S$. Then $x \in A$ for some $A \in \mathcal{F}$. Since A is open, it follows that x is an interior point of A; hence, there exists an $\epsilon > 0$ such that $N(x;\epsilon) \subseteq A$. Therefore, $N(x;\epsilon) \subseteq S$ and it follows that x is an interior point of S.

b. Let $A_1, \ldots, A_n$, where $n \in \mathbb{N}$, be a collection of opens sets and define

$$S = \bigcap_{i=1}^{n} A_i.$$

Let $x \in S$. Then, $x \in A_i$ for $i = 1, \ldots, A_n$. For each i, there is an $\epsilon_i > 0$ such that $N(x;\epsilon_i) \subseteq A_i$. Let $\epsilon = \min_i \epsilon_i$. Then, $N(x;\epsilon) \subseteq S$ and it follows that x is an interior point of S.

□

2.2 Accumulation Points

Definition 2.5. Let $S \subseteq \mathbb{R}$. We say that $x \in \mathbb{R}$ is an *accumulation point* if

$$\forall \epsilon > 0,\ N^*(x;\epsilon) \cap S \neq \emptyset.$$

The set of all accumulation points of S is denoted by S'. If $x \in S$ and $x \notin S'$, then x is called an *isolated point* of S.

Definition 2.6. Let $S \subseteq \mathbb{R}$. The *closure* of S is defined by

$$\mathrm{cl}\,(S) = S \cup S'.$$

Note that $x \in \mathrm{cl}\,(S)$ if and only if every neighborhood of x intersects S. Indeed, if $x \in \mathrm{cl}\,(S)$, then $x \in S$ or $x \in S'$, in either case every neighborhood of x intersects S. Conversely, suppose that every neighborhood of x intersects S. If $x \notin S$, then every deleted neighborhood of x intersects S, so $x \in S'$. Hence, $x \in \mathrm{cl}\,(S)$.

Theorem 2.7. *Let $S \subseteq \mathbb{R}$. Then*

a. S is closed if and only if S contains all of its accumulations point,

b. $\mathrm{cl}\,(S)$ is a closed set,

c. S is closed if and only if $S = \mathrm{cl}\,(S)$,

d. $\mathrm{cl}\,(S) = S \cup \mathrm{bd}\,(S)$.

3 Exercises

I. Prove Theorem 2.4

II. Prove Theorem 2.7

References

[1] W. Trench, *Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 2nd ed., 2013.