# Real Analysis

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## 1 Daily Quiz

Let $S = \{\frac{1}{n}\colon\ n \in \mathbb{N}\}$. What is the only accumulation point of $S$? Is $S$ compact?

## 2 Key Topics

Having laid the foundation of the real numbers, we now begin the study of sequences. For further reading, see [1, Section 2.1].

### 2.1 Sequences

**Definition 2.1.** A *sequence* is a function $s\colon \mathbb{N} \to \mathbb{R}$. We often denote the $n$th element of the sequence by $s(n) = s_n$. We may also denote the sequence by

$$(s_n)_{n=1}^{\infty} = (s_1, s_2, s_3, \ldots).$$

*Example* 2.2. a. Let $s\colon \mathbb{N} \to \mathbb{R}$ be defined by $s(n) = 1 + (-1)^n$. Then,

$$(s_n)_{n=1}^{\infty} = (0, 2, 0, \ldots).$$

b. Let $s\colon \mathbb{N} \to \mathbb{R}$ be defined by $s(n) = \frac{1}{n}$. Then,

$$(s_n)_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right).$$

c. Let $s\colon \mathbb{N} \to \mathbb{R}$ be defined by $s(n) = 2n$. Then,

$$(s_n)_{n=1}^{\infty} = (2, 4, 6, \ldots).$$

### 2.2 Convergence

**Definition 2.3.** A sequence $s\colon \mathbb{N} \to \mathbb{R}$ *converges* to $L \in \mathbb{R}$ if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{R}\ \ni\ n > N \Rightarrow |s_n - L| < \epsilon.$$

If $s$ converges to $L$, then we write

$$\lim_{n\to\infty} s_n = L$$

If the sequence does not converge to a real number, then it is said to *diverge*.

*Example* 2.4. Let $s\colon \mathbb{N} \to \mathbb{R}$ be defined by $s(n) = \frac{1}{n}$. Then, $\lim_{n\to\infty} s_n = 0$. Indeed, let $\epsilon > 0$ and define $N = \frac{1}{\epsilon}$. Then, $n > N$ gives us

$$n > \frac{1}{\epsilon} \Rightarrow \frac{1}{n} < \epsilon \Rightarrow \left|\frac{1}{n} - 0\right| < \epsilon.$$

The following theorem is helpful for analyzing the convergence of more complicated sequences.

**Theorem 2.5.** *Let $s\colon \mathbb{N} \to \mathbb{R}$ and $a\colon \mathbb{N} \to \mathbb{R}$ such that for some $L, M, k \in \mathbb{R}$, where $k > 0$,*

$$|s_n - L| \leq k\,|a_n|\,,$$

*for all $n > M$. If $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} s_n = L$.*

*Proof.* Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = 0$, there exists a $N_1 \in \mathbb{R}$ such that

$$n > \hat{N} \Rightarrow |a_n| < \frac{\epsilon}{k}.$$

Let $N = \max\{M, N_1\}$. Then, $n > N$ implies that

$$|s_n - L| \leq k\,|a_n| < k\frac{\epsilon}{k} = \epsilon.$$

□

*Example* 2.6. We will use Theorem 2.5 to prove that

$$\lim_{n\to\infty} \frac{4n^2 - 3}{5n^2 - 2n} = \frac{4}{5}.$$

Note that

$$\frac{4n^2 - 3}{5n^2 - 2n} - \frac{4}{5} = \frac{8n - 15}{25n^2 - 10n}$$

Moreover,

$$8n - 15 \leq 8n,$$

for all $n \in \mathbb{N}$, and

$$25n^2 - 10n \geq 15n^2,$$

for all $n \in \mathbb{N}$.

Let $s_n = \frac{4n^2-3}{5n^2-2n}$ and $a_n = \frac{1}{n}$. Then,

$$\left|s_n - \frac{4}{5}\right| \leq \frac{8}{15}\frac{1}{n} = \frac{8}{15}a_n,$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} a_n = 0$, it follows that $\lim_{n\to\infty} s_n = \frac{4}{5}$.

**Theorem 2.7.** *Every convergent sequence has a bounded range.*

**Theorem 2.8.** *If a sequence converges, then its limit is unique.*

## 3 Exercises

I. Prove Theorem 2.5

II. Prove Theorem 2.7

III. Prove Theorem 2.8

## References

[1] J. Lebl, *Basic Analysis: Introduction to Real Analysis*, Creative Commons Attribution-Noncommercial-Share Alike, 6th ed., 2023.