Real Analysis

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1 Daily Quiz

Define $f\colon (-\infty, 0] \to [0, \infty)$ by
$$f(x) = x^2.$$
Show that f is bijective and describe f^{-1} by a formula.

2 Key Topics

Today we introduce the cardinality of a set and use bijective functions to compare the cardinality of infinite sets. For further reading, see [1, Chapter 14].

The *cardinality* of a set is defined to be the number of elements in that set. We often denote the cardinality of a set S by $|S|$. For finite sets, cardinality is a straightforward concept; however, for infinite sets it is more subtle.

Definition 2.1. Two sets S and T have the same cardinality, or are *equinumerous*, if there exists a bijective function from S onto T. We denote equinumerous sets by $S \sim T$.

Proposition 2.2. *Let $\mathcal{F}$ be a set of sets. The relation $\sim$ is an equivalence relation on $\mathcal{F}$.*

Definition 2.3. Let $I_n = \{1, 2, \ldots, n\}$. A set S is *finite* if $S = \emptyset$ or if $I_n \sim S$ for some $n \in \mathbb{N}$. If S is not finite, then it is *infinite*.

Definition 2.4. A set S is *denumerable* if $\mathbb{N} \sim S$. If a set is finite or denumerable, then it is *countable*. If a set is not countable, then it is *uncountable*

2.1 Countable Sets

To begin, we show that $\mathbb{N}$ and $\mathbb{Z}$ are equinumerous, despite the fact that $\mathbb{N}$ is a proper subset of $\mathbb{Z}$.

Example 2.5. Define $f\colon \mathbb{N} \to \mathbb{Z}$ by the formula
$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{1-n}{2} & \text{if } n \text{ is odd.} \end{cases}$$
To show that f is injective, let $n, n' \in \mathbb{N}$ and assume that $f(n) = f(n')$. Then, n and n' must either both be even or both be odd. If n and n' are both even, then there exists $j, k \in \mathbb{N}$ such that $n = 2j$ and $n' = 2k$. Moreover, since $f(n) = f(n')$, it follows that $j = k$ and $n = n'$. If n and n' are both odd, then there exists $j, k \in \mathbb{N}$ such that $n = 2j + 1$ and $n' = 2k + 1$. Moreover, since $f(n) = f(n')$, it follows that $j = k$ and $n = n'$.

To show that f is surjective, let $m \in \mathbb{Z}$. If $m \geq 1$, then define $n \in \mathbb{N}$ as $n = 2m$. Clearly, n is even so $f(n) = \frac{n}{2} = m$. If $m \leq 0$, then define $n \in \mathbb{N}$ as $n = -(2m - 1) = 2(-m) + 1$. Clearly, n is odd so $f(n) = \frac{1-n}{2} = m$. $\square$

Next, we show that $\mathbb{Q}$ is countable using the following theorem. Note that you will prove this Theorem in Homework Assignment 2.

Theorem 2.6. *Let S be a non-empty set. The following conditions are equivalent.*

a. S is countable.

b. There exists an injection $f\colon\ S \to \mathbb{N}$.

c. There exists a surjection $g\colon\ \mathbb{N} \to S$.

Corollary 2.7. *Let S and T be non-empty countable sets. Then, $S \cup T$ is countable.*

Corollary 2.8. *Let S and T be non-empty countable sets. Then, $S \times T$ is countable.*

Corollary 2.9. *The set $\mathbb{Q}$ is countable.*

2.2 Uncountable Sets

Note that you will prove the following Theorem in Homework Assignment 2.

Theorem 2.10. *The set $\mathbb{R}$ is uncountable.*

Definition 2.11. Let be any set S. The *power set* of S, denoted by $\mathcal{P}(S)$, is the set of all subsets of S.

Theorem 2.12. *For any set S, there is no surjection $f\colon\ S \to \mathcal{P}(S)$.*

3 Exercises

I. Prove Corollary 2.7

II. Prove Corollary 2.8

III. Prove Corollary 2.9

References

[1] R. Hammack, *Book of Proof*, Creative Commons Attribution-NonCommercial-NoDerivative, 3rd ed., 2018.