Real Analysis

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1 Daily Quiz

Prove that $f\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ defined by
$$f(m,n) = 2^{m-1}(2n-1)$$
is bijective. Hint: every natural number can be written uniquely as a product of primes.

2 Key Topics

Today we finish our discussion of countable and uncountable sets and we introduce mathematical induction. For further reading, see [1, Chapters 10 and 14].

2.1 Uncountable Sets

On September 6 2023, we proved that the rational numbers were countable, i.e., there exists a bijection $f\colon\ \mathbb{N}\to\mathbb{Q}$. You will prove the following theorem in Homework Assignment 2.

Theorem 2.1. *The set $\mathbb{R}$ is uncountable.*

Now, we will use the power set to establish a spectrum of infinite cardinalities.

Definition 2.2. Let S be any set. The *power set* of S, denoted by $\mathcal{P}(S)$, is the set of all subsets of S.

Theorem 2.3. *For any set S, there is no surjection $f\colon\ S\to\mathcal{P}(S)$.*

Proof. Suppose there exists a surjection $f\colon\ S\to\mathcal{P}(S)$. Define
$$T = \{x\in S\colon\ x\notin f(x)\}.$$
Clearly $T\subseteq S$; hence, $T\in\mathcal{P}(S)$. Since f is surjective, there exists a $y\in S$ such that $T=f(y)$.

If $y\in T$, then $y\in f(y)$ which implies that $y\notin T$. If $y\notin T$, then $t\notin f(y)$ which implies that $y\in T$. Hence, we have a contradiction and it follows that f can be surjective. □

By applying Theorem 2.3 recursively, we obtain a sequence of infinite cardinalities:
$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots.$$

2.2 Mathematical Induction

Axiom 2.4 (Well-Ordering of $\mathbb{N}$)**.** Every non-empty subset $S\subseteq\mathbb{N}$ has an element $m\in S$ such that $m\le k$ for all $k\in S$.

Theorem 2.5 (Principle of Mathematical Induction)**.** *Let $P(n)$ be a statement that is either true or false for each $n\in\mathbb{N}$. Then, $P(n)$ is true for all $n\in\mathbb{N}$ provided that*

a. $P(1)$ is true and

b. For each $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

Proof. For the sake of contradiction, suppose that a. and b. hold but $P(n)$ is false for some $n \in \mathbb{N}$. Define

$$S = \{n \in \mathbb{N}\colon\ P(n) \text{ is false}\}.$$

Then, S is a non-empty subset of $\mathbb{N}$. So, the well-ordering property of $\mathbb{N}$ states that there exists a $m \in S$ such that $m \leq k$ for all $k \in S$. By a., $m > 1$. Therefore, $(m-1) \in \mathbb{N}$ and $(m-1) \notin S$. So, b. implies that $P(m)$ is true which is a contradiction. □

Example 2.6. We will use mathematical induction to prove that

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1),\ \forall n \in \mathbb{N}.$$

The base case is true since

$$1 = \frac{1}{2}(1)(1+1).$$

Now, let $k \in \mathbb{N}$ and assume that

$$1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1).$$

Then,

$$\begin{aligned}
1 + 2 + 3 + \cdots + k + (k+1) &= \frac{1}{2}k(k+1) + (k+1) \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{k^2 + 3k + 2}{2} \\
&= \frac{1}{2}(k+1)(k+2)
\end{aligned}$$

3 Exercises

I. Prove Theorem 2.3

II. Prove Theorem 2.5

III. Perform Example 2.6

References

[1] R. Hammack, *Book of Proof*, Creative Commons Attribution-NonCommercial-NoDerivative, 3rd ed., 2018.