

Natural Numbers, Integers, and Rational Numbers Quiz

Math 312: Concepts of Real Analysis

Name: _____

Date: _____

Directions. Show enough work to support each answer. You may use your class notes. No electronic devices are permitted.

1. The construction and algebra of $\mathbb{N}$.

Recall that $0 = \emptyset$, $S(n) = n \cup \{n\}$, and

$$m + 0 = m, \quad m + S(n) = S(m + n),$$

$$m \cdot 0 = 0, \quad m \cdot S(n) = m \cdot n + m.$$

- (a) Write 1, 2, and 3 explicitly as sets using only braces and $\emptyset$. Recall that $m < n$ means $m \in n$. Use your sets to explain why $2 < 3$.
- (b) Use the recursive definitions to evaluate $2 \cdot 2$. Show each recursive step.
- (c) Prove by induction that $0 + n = n$ for every $n \in \mathbb{N}$.

2. **The constructions of $\mathbb{Z}$ and $\mathbb{Q}$.**

- (a) The integers are equivalence classes of pairs in $\mathbb{N} \times \mathbb{N}$. Complete the definition

$$(a, b) \sim (c, d) \iff \text{_____},$$

and use it to determine whether $(5, 2) \sim (7, 4)$.

- (b) Using

$$[(a, b)] + [(c, d)] = [(a + c, b + d)],$$

find the additive inverse of $[(3, 1)]$ and verify that your answer adds to the zero element of $\mathbb{Z}$.

- (c) The rational numbers are equivalence classes of pairs in $\mathbb{Z} \times \mathbb{N}$. State the equivalence relation and the addition rule for these equivalence classes, and then compute

$$[(1, 2)] + [(1, 3)].$$

3. Order, density, and the rational metric.

- (a) Let $p, q \in \mathbb{Q}$ with $p < q$. Define $r = (p + q)/2$. Explain why $r \in \mathbb{Q}$ and prove that

$$p < r < q.$$

What property of $\mathbb{Q}$ does this establish?

- (b) The usual metric on $\mathbb{Q}$ is $d(p, q) = |p - q|$. Compute

$$d\left(-\frac{2}{3}, \frac{5}{6}\right).$$

- (c) State precisely what it means for a sequence (x_n) in $\mathbb{Q}$ to be a Cauchy sequence with respect to this metric.