

Natural Numbers, Integers, and Rational Numbers Quiz

Math 312: Concepts of Real Analysis

Answer Key

1. The construction and algebra of $\mathbb{N}$.

Recall that $0 = \emptyset$, $S(n) = n \cup \{n\}$, and

$$m + 0 = m, \quad m + S(n) = S(m + n),$$

$$m \cdot 0 = 0, \quad m \cdot S(n) = m \cdot n + m.$$

- (a) Write 1, 2, and 3 explicitly as sets using only braces and $\emptyset$. Recall that $m < n$ means $m \in n$. Use your sets to explain why $2 < 3$.
- (b) Use the recursive definitions to evaluate $2 \cdot 2$. Show each recursive step.
- (c) Prove by induction that $0 + n = n$ for every $n \in \mathbb{N}$.

Solution. We have

$$1 = \{\emptyset\}, \quad 2 = \{\emptyset, \{\emptyset\}\},$$

and

$$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}.$$

Since $2 \in 3$, our definition of order gives $2 < 3$.

Since $2 = S(S(0))$,

$$2 \cdot 2 = 2 \cdot S(S(0)) = 2 \cdot S(0) + 2 = (2 \cdot 0 + 2) + 2 = (0 + 2) + 2 = 4.$$

Let $P(n)$ be the statement $0 + n = n$. The base case is $0 + 0 = 0$. Suppose $0 + n = n$. Then

$$0 + S(n) = S(0 + n) = S(n).$$

Thus, $0 + n = n$ for every $n \in \mathbb{N}$ by induction.

2. The constructions of $\mathbb{Z}$ and $\mathbb{Q}$.

- (a) The integers are equivalence classes of pairs in $\mathbb{N} \times \mathbb{N}$. Complete the definition

$$(a, b) \sim (c, d) \iff \text{_____},$$

and use it to determine whether $(5, 2) \sim (7, 4)$.

- (b) Using

$$[(a, b)] + [(c, d)] = [(a + c, b + d)],$$

find the additive inverse of $[(3, 1)]$ and verify that your answer adds to the zero element of $\mathbb{Z}$.

- (c) The rational numbers are equivalence classes of pairs in $\mathbb{Z} \times \mathbb{N}$. State the equivalence relation and the addition rule for these equivalence classes, and then compute

$$[(1, 2)] + [(1, 3)].$$

Solution. For the construction of $\mathbb{Z}$,

$$(a, b) \sim (c, d) \iff a + d = b + c.$$

Since $5 + 4 = 2 + 7 = 9$, we have $(5, 2) \sim (7, 4)$.

The additive inverse of $[(3, 1)]$ is $[(1, 3)]$. Indeed,

$$[(3, 1)] + [(1, 3)] = [(4, 4)] = [(0, 0)],$$

which is the zero element of $\mathbb{Z}$.

For the construction of $\mathbb{Q}$,

$$(a, b) \sim (c, d) \iff ad = bc.$$

Using $[(a, b)] + [(c, d)] = [(ad + bc, bd)]$, we obtain

$$[(1, 2)] + [(1, 3)] = [(3 + 2, 6)] = [(5, 6)].$$

3. Order, density, and the rational metric.

(a) Let $p, q \in \mathbb{Q}$ with $p < q$. Define $r = (p + q)/2$. Explain why $r \in \mathbb{Q}$ and prove that

$$p < r < q.$$

What property of $\mathbb{Q}$ does this establish?

(b) The usual metric on $\mathbb{Q}$ is $d(p, q) = |p - q|$. Compute

$$d\left(-\frac{2}{3}, \frac{5}{6}\right).$$

(c) State precisely what it means for a sequence (x_n) in $\mathbb{Q}$ to be a Cauchy sequence with respect to this metric.

Solution. Since $\mathbb{Q}$ is a field, it is closed under addition and division by the nonzero rational number 2, so $r = (p + q)/2 \in \mathbb{Q}$. From $p < q$, adding p gives $2p < p + q$, and division by the positive number 2 gives

$$p < \frac{p + q}{2} = r.$$

Similarly, adding q to $p < q$ gives $p + q < 2q$, so $r < q$. Thus, there is a rational number strictly between every two distinct rational numbers; this is the density of $\mathbb{Q}$.

The distance is

$$d\left(-\frac{2}{3}, \frac{5}{6}\right) = \left|-\frac{2}{3} - \frac{5}{6}\right| = \left|-\frac{9}{6}\right| = \frac{3}{2}.$$

A sequence (x_n) in $\mathbb{Q}$ is Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$d(x_m, x_n) = |x_m - x_n| < \varepsilon.$$