Power Series

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1 Power Series

Let $a\colon \mathbb{N} \to \mathbb{R}$ be a sequence of real numbers and let $x_0 \in \mathbb{R}$. The series

$$\sum_{k=0}^{\infty} a_k(x-x_0)^k = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots,$$

is called a *power series*. The number a_k is called the kth *coefficient* of the series, and the number x_0 is the *center* of the series.

The *radius of convergence* of the power series is a value $R \geq 0$ such that the power series converges absolutely for all $x \in (x_0 - R, x_0 + R)$, and diverges for all $x \in \mathbb{R}$ such that $|x - x_0| > R$. The following theorem uses the ratio test to determine the radius of convergence of a power series.

Theorem 1.1. *Let $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series and let $\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$, provided the limit exists. Then, the radius of convergence of the power series is*

$$R = \begin{cases} \frac{1}{\rho} & \text{if } 0 < \rho < \infty \\ \infty & \text{if } \rho = 0 \\ 0 & \text{if } \rho = \infty \end{cases}$$

Proof. By the ratio test, the series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ converges absolutely if

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| |x-x_0| = \rho\,|x-x_0| < 1,$$

and diverges if $\rho\,|x-x_0| > 1$. If $\rho = \infty$, then we have absolute convergence only when $x = x_0$. If $\rho = 0$, then we have absolute convergence for all $x \in \mathbb{R}$. If $0 < \rho < \infty$, then we have absolute convergence when

$$|x - x_0| < \frac{1}{\rho},$$

and divergence when $|x - x_0| > \frac{1}{\rho}$. □

The *interval of convergence* of the power series is the set of all x such that the series converges. The value x_0 is always in the interval of convergence. If R is positive, then $(x_0 - R, x_0 + R)$ is a subset of the interval of convergence. If R is finite and non-zero, then the end points $x_0 \pm R$ may be included in the interval of convergence. For example, consider the power series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k.$$

The radius of convergence of this series is $R = 1$ and its interval of convergence is $(0, 2]$.

2 Pointwise vs Uniform Convergence

Let $(f_n)_{n=0}^{\infty}$ be a sequence of functions defined on a subset $S \subseteq \mathbb{R}$. This sequence *converges pointwise* on S if for each $x \in S$, the sequence of numbers $(f_n(x))_{n=0}^{\infty}$ converges. In this case, we define $f\colon S \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

For example, consider the sequence of functions defined by $f_n(x) = x^n$ on $[0,1]$. Then, for $x \in [0,1)$, $\lim_{n\to\infty} x^n = 0$; for $x = 1$, we have $\lim_{n\to\infty} f_n(1) = 1$. Thus, the sequence $f_n(x) = x^n$ converges pointwise on $[0,1]$; furthermore, the limit function is given by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Note that each $f_n(x)$ is continuous on $[0,1]$; however, $f(x)$ is not continuous at $x = 1$. In terms of limit operations, we have

$$\lim_{x\to 1^-} \left(\lim_{n\to\infty} f_n(x) \right) = \lim_{x\to 1^-} (0) = 0$$

whereas

$$\lim_{n\to\infty} \left(\lim_{x\to 1^-} f_n(x) \right) = \lim_{n\to\infty} (1) = 1.$$

Let $(f_n)_{n=0}^{\infty}$ be a sequence of functions defined on a subset $S \subseteq \mathbb{R}$. This sequence *converges uniformly* on S to a function $f\colon S \to \mathbb{R}$ if for all $\epsilon \in \mathbb{R}_{>0}$, there is a $N \in \mathbb{N}$ such that

$$n \ge N \Rightarrow |f_n(x) - f(x)| < \epsilon,$$

for all $x \in S$. It is important to clarify the distinction between pointwise and uniform convergence. With pointwise convergence, for each $x \in S$, there is a $f(x) \in \mathbb{R}$ such that for all $\epsilon \in \mathbb{R}_{>0}$, there is a $N \in \mathbb{N}$ such that

$$n \ge N \Rightarrow |f_n(x) - f(x)| < \epsilon.$$

In this case, the N may depend on both x and ϵ. With uniform convergence, there exists a function $f\colon S \to \mathbb{R}$ such that for all $\epsilon \in \mathbb{R}_{>0}$, there is a $N \in \mathbb{N}$ such that

$$n \ge N \Rightarrow |f_n(x) - f(x)| < \epsilon,$$

for all $x \in S$. In this case, the N only depends on ϵ.

To illustrate the difference between pointwise and uniform convergence, consider the sequence of functions defined by $f_n(x) = x^n$ on $[0,1]$. We have seen that this sequence converges pointwise to the function $f\colon S \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

However, if we fix $\epsilon = 1/2$, then for all $n \in \mathbb{N}$

$$2^{-1/n} \le x < 1 \Rightarrow f_n(x) \ge 1/2.$$

Therefore, for all $n \in \mathbb{N}$, there is an $x \in [0,1)$ such that

$$|f_n(x) - f(x)| = f_n(x) \ge 1/2 = \epsilon.$$

Thus, this sequence does not converge uniformly.

Now, let $t \in (0,1)$. Then, for any $\epsilon \in \mathbb{R}_{>0}$, there is a $N \in \mathbb{N}$ such that

$$t^N < \epsilon.$$

So, for all $n \ge N$, $t^n < \epsilon$. Therefore, for all $n \ge N$,

$$|f_n(x) - 0| = x^n \le t^n < \epsilon,$$

for all $x \in [0,t]$. Thus, the sequence $f_n(x) = x^n$ converges uniformly to $f(x) = 0$ on $[0,t]$.

3 Applications of Uniform Convergence

Theorem 3.1. *Let* $(f_n)_{n=0}^{\infty}$ *be a sequence of continuous functions defined on a set* $S \subset \mathbb{R}$ *that converges uniformly to* $f\colon S \to \mathbb{R}$*. Then,* f *is continuous on* S*.*

Theorem 3.2. *Let* $(f_n)_{n=0}^{\infty}$ *be a sequence of Riemann integrable functions defined on the interval* $[a,b]$ *that converges uniformly to* $f\colon [a,b] \to \mathbb{R}$*. Then,* f *is Riemann integrable and*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

Theorem 3.3. *Let* $(f_n)_{n=0}^{\infty}$ *be a sequence of differentiable functions defined on the interval* $[a,b]$ *that converges to* $f\colon [a,b] \to \mathbb{R}$*. If* (f_n') *converges uniformly on* $[a,b]$*, then* f *is differentiable and*

$$f'(x) = \lim_{n\to\infty} f_n'(x),$$

for all $x \in [a,b]$*.*

4 Uniform Convergence of Power Series

Theorem 4.1. *Suppose the power series* $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ *has radius of convergence* R*, where* $0 < R \leq \infty$*. Let* $f_n(x) = \sum_{k=0}^{n} a_k(x-x_0)^k$ *and* $0 < K < R$*. Then,* $f_n(x)$ *converges uniformly on* $[x_0 - K, x_0 + K]$*.*

Theorem 4.2. *Suppose the power series* $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ *has radius of convergence* R*, where* $0 < R \leq \infty$*. Let* $f_n(x) = \sum_{k=0}^{n} a_k(x-x_0)^k$*. If the power series converges at* $x_0 + R$*, then* $f_n(x)$ *converges uniformly on* $[x_0, x_0+R]$*. Similarly, if the power seres converges at* $x_0 - R$*, then* $f_n(x)$ *converges uniformly on* $[x_0 - R, x_0]$*.*